

The $1/N_c$ Expansion for Baryons

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Abstract

A systematic expansion in $1/N_c$ is constructed for baryons in QCD. Predictions of the $1/N_c$ expansion at leading and subleading order for baryon axial current coupling constant ratios such as F/D , baryon masses and magnetic moments are derived. The baryon sector of QCD has a light quark spin-flavor symmetry at leading order in $1/N_c$. The formalism of induced representations for contracted Lie algebras is introduced to explain the consequences of this symmetry. Relations are first derived for the simplest case of two light flavors of quarks. The generalization of the large N_c expansion to $N_f = 3$ flavors is subtle and is treated using several complementary methods. The $1/N_c$ expansion severely restricts the form of $SU(3)$ breaking in the baryon sector. Extrapolation of the large N_c results to $N_c = 3$ permits quantitative comparison with experimental data. Deviations of the measured quantities from exact spin-flavor symmetry predictions are accurately described by the subleading $1/N_c$ corrections. Implications of the $1/N_c$ expansion for baryon chiral perturbation theory are discussed.

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1. Introduction

The theory of the strong interactions is a strongly coupled theory at low energies, with no small expansion parameter. The absence of a small expansion parameter has frustrated attempts to compute low-energy properties of hadrons directly in QCD. 't Hooft realized that QCD has a hidden parameter, N_c , the number of colors, and that the theory simplifies in the $N_c \rightarrow \infty$ limit [1]. In the large N_c limit, the meson sector of QCD consists of a spectrum of narrow resonances, and meson-meson scattering amplitudes are suppressed by powers of $1/\sqrt{N_c}$. The analysis of the baryon sector of QCD in the large N_c limit is more subtle because a baryon is a confined state of N_c quarks, and becomes a bound state of an infinite number of quarks when $N_c \rightarrow \infty$. Witten analyzed the interactions of baryons with mesons in large N_c [2], and showed that the N_c dependence of the baryon-meson amplitudes was the same as in a semiclassical soliton model with coupling constant $1/\sqrt{N_c}$. The Skyrme model [3], in which the baryon is a soliton of the low-energy chiral Lagrangian, is an explicit realization of Witten's idea that the baryon is a semiclassical soliton.

Although the large N_c limit of QCD was originally proposed as a quantitative calculational method, predictions of this approach remained largely qualitative in nature, with most results following primarily from large N_c power counting arguments. Recent work [4][5] on the low-energy pion interactions of baryons in large N_c shows that the large N_c expansion of QCD makes definite quantitative predictions for the static properties of baryons. The $N_c \rightarrow \infty$ predictions of QCD satisfy light quark spin-flavor symmetry relations. These symmetry relations are the same as those obtained in the large N_c Skyrme [6][7] and non-relativistic quark models [8], which yield identical group theoretic results in the large N_c limit [9]. The leading deviations from spin-flavor symmetry relations at $N_c \rightarrow \infty$ are parametrized by $1/N_c$ -suppressed operators. It is the inclusion of $1/N_c$ -suppressed effects which enables a quantitative comparison of the predictions of the large N_c expansion with the physical situation of $N_c = 3$. Whether the $1/N_c$ expansion proves useful depends on the size of the $1/N_c$ corrections. The $1/N_c$ expansion is particularly good for certain static quantities such as the baryon-pion coupling constants and the isovector magnetic moments, because there are no $1/N_c$ corrections in QCD to the spin-flavor symmetry predictions [4]. Thus, the large N_c predictions for the ratios of the baryon-pion couplings and the isovector magnetic moments are valid up to corrections of order $1/N_c^2$. The phenomenological success of light quark spin-flavor symmetry predictions for these quantities is explained by the $1/N_c$ expansion.

There are several assumptions implicit in the large N_c approach to QCD. The principal assumption is that certain properties of QCD, such as confinement and chiral symmetry breaking [10], persist as N_c is taken to infinity. The confined hadronic states of the large N_c theory are mesons and baryons, and the lowest-lying hadrons are the pseudo-Goldstone bosons of spontaneous chiral symmetry breaking, and baryons with N_c quarks. The results derived in this paper do not require chiral symmetry to be exact, and are valid even for non-zero quark masses. The only requirement on the quark masses is that they remain finite as $N_c \rightarrow \infty$. We will assume isospin symmetry is exact, for simplicity. Given these assumptions, the principal interactions of baryons at low energies are pion interactions. Since the baryon is a coherent state of N_c quarks, and the pion couples to each of these quarks, the pion-baryon axial vector coupling constant is of order N_c . Each single pion-baryon vertex is suppressed by one factor of f_π , which grows as $\sqrt{N_c}$ in the large N_c limit. Thus, the pion-baryon vertex is of order $\sqrt{N_c}$. Baryon-pion scattering amplitudes involve two baryon-pion vertices and therefore will grow like N_c . This large N_c behavior of the baryon-pion scattering amplitude violates unitarity unless there is a cancellation amongst diagrams with different intermediate baryon states. Thus, consistency of the large N_c limit results in cancellation conditions which relate different pion-baryon coupling constants. These constraint equations imply that the baryon sector of QCD possesses a contracted light quark spin-flavor symmetry in the large N_c limit [4][11][12]. The contracted spin-flavor symmetry requires that the baryon sector of large N_c QCD contains an infinite tower of degenerate states with $I = J = 1/2, 3/2, 5/2, \dots$, and with pion couplings in the precise ratios as those given by the large N_c Skyrme and non-relativistic quark models.

In this paper, we give a detailed analysis of the predictions of the $1/N_c$ expansion for baryons, and we provide details of some computations referred to in the previous work [4][5]. The previous results were derived in a straightforward (but tedious) manner using Clebsch-Gordan coefficients. In this paper, they are rederived using more elegant group theoretical methods from the theory of induced representations. This formalism makes the comparison of large N_c QCD with the Skyrme model more transparent. The results for two light flavors are extended to baryons containing strange quarks. Some of the results derived for baryons with strange quarks are obtained using large N_c *but without using $SU(3)$ symmetry*. These results hold irrespective of the mass of the s -quark, and provide strong constraints on the pattern of $SU(3)$ breaking in the baryon sector. The methods discussed in this work also apply to baryons containing a single heavy quark in the $m_Q \rightarrow \infty$ limit.

The $N_c \rightarrow \infty$ constraint equations for the baryon-pion couplings are identical to equations derived a long time ago in the study of strong-coupling models. The logic of the $1/N_c$ expansion presented here is quite different from these other lines of reasoning, however. Many of the old derivations predate QCD or use arguments which are not justified in QCD. The earliest work by Pauli and Dancoff [13] showed that a static $I = J = 1/2$ nucleon strongly coupled to a p -wave pion produces an infinite tower of baryon states with $I = J = 1/2, 3/2, 5/2, \dots$, which is precisely the spectrum of large N_c QCD. The constraint equations on the pion-couplings found in large N_c are the bootstrap equations of Chew [14] derived using Chew-Low theory [15]. The spectrum and properties of the strong coupling model were studied extensively by Goebel [16], and by Cook and Sakita [17] and the reader is referred to these papers for additional references to the earlier literature. The constraint equations are also similar to some results of Weinberg [18] derived by studying the high-energy behavior of scattering amplitudes.

The organization of this paper is as follows. Sect. 2 derives the contracted $SU(4)$ spin-flavor algebra for baryons of large N_c QCD using unitarity constraints on pion-baryon scattering. Sect. 3 explains the standard mathematical technique of induced representations which is used to construct irreducible representations of the spin-flavor algebra for baryons. Sect. 4 makes explicit the connection between the induced representations of Sect. 3 and the standard collective coordinate quantization of Skyrmions. Sect. 5 discusses the large N_c counting rules for baryons. The identification of the QCD baryons with particular irreducible representations is justified in this section. Sect. 6 derives the pion-baryon couplings in QCD, including $1/N_c$ corrections. The results are derived for baryons containing u , d or s quarks without assuming $SU(3)$ symmetry. Sect. 7 derives the kaon-baryon couplings at leading order in $1/N_c$, and Sect. 8 derives the η couplings. The meson-baryon couplings in the $SU(3)$ limit are discussed in Sect. 9. It is shown that the F/D ratio for the axial currents and baryon magnetic moments is $2/3$, up to corrections of order $1/N_c^2$. Sect. 10 discusses the baryon masses, including $1/N_c$ corrections, but without assuming $SU(3)$ symmetry. The extension of the theory of induced representations to exact $SU(3)$ flavor symmetry is discussed in Sect. 11. The implications of the $1/N_c$ expansion for chiral perturbation theory are presented in Sect. 12. The conclusions are given in Sect. 13. Since the paper is rather long, we summarize the main results below.

1.1. Summary of Main Results

- The F/D ratio for the baryon axial currents is determined to be $2/3 + \mathcal{O}(1/N_c^2)$, in good agreement with the experimental value of 0.58 ± 0.04 .
- The F/D ratio for the baryon magnetic moments is determined to be $2/3 + \mathcal{O}(1/N_c^2)$, in good agreement with the experimental value of 0.72. The difference between the F/D ratios for the axial currents and magnetic moments is an indication of the size of $1/N_c^2$ corrections.
- The ratios of all pion-baryon couplings are determined up to corrections of order $1/N_c^2$, and the ratios of all kaon-baryon couplings are determined to leading order. These results are independent of the mass of the s -quark.
- The $SU(3)$ breaking in the pion couplings must be linear in strangeness at order $1/N_c$. This leads to an equal spacing rule for the pion couplings, which agrees well with the data. The $SU(3)$ breaking in the decuplet-octet transition axial currents is related to the $SU(3)$ breaking in the octet axial currents.
- The baryon mass relations

$$\begin{aligned}
\Sigma^* - \Sigma &= \Xi^* - \Xi \\
\frac{1}{3}(\Sigma + 2\Sigma^*) - \Lambda &= \frac{2}{3}(\Delta - N) \\
\frac{3}{4}\Lambda + \frac{1}{4}\Sigma - \frac{1}{2}(N + \Xi) &= -\frac{1}{4}(\Omega - \Xi^* - \Sigma^* + \Delta) \\
\frac{1}{2}(\Sigma^* - \Delta) - (\Xi^* - \Sigma^*) + \frac{1}{2}(\Omega - \Xi^*) &= 0 \\
\Sigma_Q^* - \Sigma_Q &= \Xi_Q^* - \Xi_Q' \\
\frac{1}{3}(2\Sigma_Q^* + \Sigma_Q) - \Lambda_Q &= \frac{2}{3}(\Delta - N)
\end{aligned}$$

are valid up to corrections of order $1/N_c^2$ without assuming $SU(3)$ symmetry. Some of these relations are also valid using broken $SU(3)$, with octet symmetry breaking. Relations which can be proved using either large N_c or $SU(3)$ work extremely well, because effects which violate these relations must break both symmetries. The implications of $SU(3)$ breaking for the above relations are discussed more quantitatively in Sect. 10.

- The chiral loop correction to the baryon axial currents cancels to two orders in the $1/N_c$ expansion, and is of order $1/N_c$ instead of order N_c .
- The order N_c non-analytic correction to the baryon masses is pure $SU(3)$ singlet, and the order one contribution is pure $SU(3)$ octet. Thus violations of the Gell-Mann–Okubo formula are at most order $1/N_c$. The baryon masses can be strongly non-linear functions of the strange quark mass, and still satisfy the Gell-Mann–Okubo formula.

2. The Spin-Flavor Algebra for Baryons in Large N_c QCD

An effective light quark spin-flavor symmetry for baryons emerges in the large N_c limit of QCD. In this section, the spin-flavor algebra for baryons in the large N_c limit is derived for the case of $N_f = 2$ light quark flavors. The generalization of this symmetry to three light flavors is subtle, and is addressed in later sections. For $N_f = 2$ light flavors, large N_c baryon representations occur with the same spin and isospin quantum numbers as QCD $N_c = 3$ baryon representations. For $N_f = 3$ light flavors, the $SU(3)$ flavor representations of large N_c baryons are different from $N_c = 3$ baryons. Hence, the identification of baryon states in the large N_c limit with the physical baryon states of $N_c = 3$ QCD is not unique. This ambiguity complicates the discussion for $N_f = 3$. The derivation of the contracted spin-flavor symmetry for baryons of this section assumes that the baryon mass is of order N_c , the axial vector coupling constant g_A is of order N_c , and the pion decay constant f_π is of order $\sqrt{N_c}$ in the large N_c limit. These assumptions are justified in Sect. 5.

The spin-flavor algebra of large N_c baryons is derived by studying the interactions of baryons with low-energy pions. In the large N_c limit, the baryon mass is order N_c , so a baryon is infinitely heavy compared to a pion. Thus, the baryon can be treated as a static fermion, and the pion-baryon coupling can be analyzed in the rest frame of the baryon. Since pions are pseudo-Goldstone bosons of chiral symmetry breaking, they are derivatively coupled to the axial vector baryon current. A general pion-baryon coupling is written as the baryon axial current matrix element

$$\langle B' | \bar{q} \gamma^i \gamma_5 \tau^a q | B \rangle = N_c g (X^{ia})_{B'B}, \quad (2.1)$$

times the derivatively coupled pion field $\partial^i \pi^a / f_\pi$, where the index a represents the flavor index of the pion and the index i is a spin index. The operator X^{ia} has an expansion in powers of $1/N_c$, $X^{ia} = X_0^{ia} + X_1^{ia}/N_c + \dots$. The p -wave pion carries spin one and isospin one. The labels B and B' include the spin and isospin quantum numbers of the baryons. The axial vector matrix element (2.1) is non-vanishing only for baryon states B and B' with spin and isospin quantum numbers combined to form a spin one and isospin one axial vector current. In addition, the baryon states must be degenerate in the large N_c limit, since a soft pion will not access baryon states separated by a mass gap. In eq. (2.1), the coupling constant g is chosen so that the matrix X_0^{ia} has a convenient normalization, which will be chosen later. An explicit factor of N_c is factored out of the matrix element to keep all N_c -dependence manifest. Since $f_\pi \sim \sqrt{N_c}$ in the large N_c limit, the baryon-pion vertex

grows as $\sqrt{N_c}$. The growth of the baryon-pion vertex as $\sqrt{N_c}$ in the large N_c limit results in consistency conditions for the baryon-pion matrix elements.

Consistency conditions for pion-baryon coupling constants can be derived by looking at the large N_c behavior of pion-baryon scattering. Consider the scattering amplitude for the process $\pi^a(\omega, \mathbf{k}) + B \rightarrow \pi^b(\omega, \mathbf{k}') + B'$ in the $N_c \rightarrow \infty$ limit at fixed pion energy ω . The dominant diagrams in the large N_c limit are shown in fig. 1. The scattering amplitude is given by

$$\mathcal{A} = -i \frac{N_c^2 g^2}{f_\pi^2} \frac{k^i k'^j}{\omega} \left[X_0^{jb}, X_0^{ia} \right]_{B'B}, \quad (2.2)$$

where the matrix product of the X_0 's sums over all possible baryon intermediate states. The commutator $\left[X_0^{jb}, X_0^{ia} \right]$ arises from a relative minus sign between the two graphs in fig. 1 because the intermediate baryon in fig. 1(a) is off-shell by an energy ω , whereas the intermediate baryon in fig. 1(b) is off-shell by an energy $-\omega$. The incident and emitted pion have the same energy, since no energy can be transferred to an infinitely heavy baryon. Only intermediate states which are degenerate with the initial and final baryons in the $N_c \rightarrow \infty$ limit contribute to the amplitude. The pion-baryon scattering amplitude, which is in a single partial wave, grows as N_c in violation of unitarity, unless the pion-baryon couplings satisfy the consistency condition [4][11][14]

$$\left[X_0^{jb}, X_0^{ia} \right] = 0. \quad (2.3)$$

Since X_0^{ia} is an irreducible tensor operator with spin one and isospin one, it satisfies the commutation relations

$$\left[J^i, X_0^{jb} \right] = i \epsilon_{ijk} X_0^{kb}, \quad \left[I^a, X_0^{jb} \right] = i \epsilon_{abc} X_0^{jc}, \quad (2.4)$$

where J^i and I^a are generators of spin and isospin transformations, respectively. These generators satisfy the usual commutation relations for spin and isospin,

$$\left[J^i, J^j \right] = i \epsilon_{ijk} J^k, \quad \left[I^a, I^b \right] = i \epsilon_{abc} I^c, \quad \left[I^a, J^i \right] = 0. \quad (2.5)$$

Eqs. (2.3), (2.4) and (2.5) are the commutation relations of a contracted $SU(4)$ algebra [4][11][12][16]. Consider the embedding of the spin \otimes flavor group $SU(2) \otimes SU(2)$ in a larger $SU(4)$ group such that the defining representation $\mathbf{4} \rightarrow (\mathbf{2}, \mathbf{2})$ under the decomposition $SU(4) \rightarrow SU(2) \otimes SU(2)$.[†] If the generators of $SU(2) \otimes SU(2)$ in the defining

[†] $SU(2)$ representations will be labeled either by their J value, or by their dimension $\mathbf{2J} + \mathbf{1}$ in boldface.

representation are J^i and I^a , the $SU(4)$ generators in the defining representation are proportional to $J^i \otimes 1$, $1 \otimes I^a$ and $J^i \otimes I^a$, which will be denoted by I^a , J^i and G^{ia} , respectively. (The properly normalized $SU(4)$ generators are $I^a/\sqrt{2}$, $J^i/\sqrt{2}$ and $\sqrt{2} G^{ia}$.) The commutation relations of $SU(4)$ are

$$\begin{aligned} [J^i, J^j] &= i \epsilon_{ijk} J^k, & [I^a, I^b] &= i \epsilon_{abc} I^c, \\ [I^a, G^{jb}] &= i \epsilon_{abc} G^{jc}, & [J^i, G^{jb}] &= i \epsilon_{ijk} G^{kb}, \\ [I^a, J^i] &= 0, & [G^{ia}, G^{jb}] &= \frac{i}{4} \epsilon_{ijk} \delta_{ab} J^k + \frac{i}{4} \epsilon_{abc} \delta_{ij} I^c. \end{aligned} \quad (2.6)$$

The large N_c spin-flavor algebra for baryons is obtained by taking the limit

$$X_0^{ia} = \lim_{N_c \rightarrow \infty} \frac{G^{ia}}{N_c}. \quad (2.7)$$

This limiting procedure is known as a contraction. The only $SU(4)$ commutation relation which is affected by the contraction is $[G^{ia}, G^{jb}]$ which is not homogeneous in G , and turns into the consistency condition for pion-baryon scattering eq. (2.3). The other commutation relations become eqs. (2.4) and (2.5).

A better understanding of this contracted spin-flavor Lie algebra is obtained by considering other examples of group contractions. A simple example of group contractions is provided by the rotation group with commutation relations

$$[J_1, J_2] = i J_3, \quad [J_2, J_3] = i J_1, \quad [J_3, J_1] = i J_2. \quad (2.8)$$

One possible contraction is to define $X_i = J_i/\lambda$, and take the limit $\lambda \rightarrow \infty$. This leads to the trivial Abelian algebra $[X_i, X_j] = 0$. A more interesting contraction is to define $P_1 = J_1/\lambda$, $P_2 = J_2/\lambda$, and take the limit $\lambda \rightarrow \infty$, without rescaling J_3 . This contraction gives the algebra

$$[P_1, P_2] = 0, \quad [J_3, P_2] = -i P_1, \quad [J_3, P_1] = i P_2, \quad (2.9)$$

which is the Lie algebra of motions in the $x_1 - x_2$ plane where J_3 generates rotations about the x_3 axis, and P_1 and P_2 generate translations along the x_1 and x_2 axes. This algebra is obtained from the rotation group which describes the motion of a point on the two-sphere by considering a neighborhood of the north pole of the sphere, of size $1/\lambda$, *i.e.* points (x_1, x_2, x_3) where $x_3 \approx 1$ and x_1 and x_2 are of order $1/\lambda$. Look at this region under a magnifying glass, so that it is enlarged back to finite size. This magnification is equivalent to using coordinates near the north pole of the form $(x_1, x_2, x_3) = (y_1/\lambda, y_2/\lambda, 1)$,

and labeling the points by (y_1, y_2) . The generators of motions on the sphere in the new coordinates are now J_3 and J_1/λ and J_2/λ . In the limit that $\lambda \rightarrow \infty$, the neighborhood of the north pole becomes flat. The generator J_3 in this limit still generates rotations about the x_3 axis, and the generators J_1 and J_2 generate translations in y_1 and y_2 .

The physical interpretation of the contracted spin-flavor algebra for baryons can be made in analogy to the above example. In the large N_c limit, the axial vector matrix elements of the baryon fields become classical objects since the contributions of quarks in the baryon add coherently, and the axial currents grow with N_c . The operators X_0^{ia} which are the rescaled axial currents have a well-defined large N_c limit in which they become classical commuting variables. The normal spin and isospin symmetries of the baryon states are thus extended to include the spin-flavor generators X_0^{ia} . This extension is possible because the baryon field is static in the large N_c limit.

The large N_c consistency condition for the pion-baryon couplings eq. (2.3) determines the matrix elements up to an overall scale. In ref. [4], this solution was found by writing X_0^{ia} in terms of reduced matrix elements times Clebsch-Gordan coefficients, and then solving the consistency conditions for the reduced matrix elements. The same results can also be obtained by classifying all the irreducible representations of the contracted $SU(4)$ algebra. This method is pursued in the next section using the theory of induced representations. The construction of induced representations is closely related to the quantization of Skyrmsions. The connection between large N_c QCD and the Skyrme model is discussed in Sect. 4.

3. Induced Representations

The theory of induced group representations gives a complete classification of all irreducible representations of a semidirect product $\mathcal{G} \ltimes \mathcal{A}$ of a compact Lie group \mathcal{G} and an Abelian group \mathcal{A} . The contracted $SU(4)$ spin-flavor algebra for baryons in large N_c QCD is the semidirect product of an $SU(2) \otimes SU(2)$ Lie algebra generated by J^i and I^a , and an Abelian algebra generated by X_0^{ia} . The irreducible representations of the contracted spin-flavor algebra contain the baryon representations of large N_c QCD. In this section, all possible irreducible representations of the contracted spin-flavor algebra are constructed, and the representations which describe large N_c baryons are identified. Much of the discussion is well known [17][19], but may not be familiar to most physicists.

The irreducible representations of a semidirect product $\mathcal{G} \ltimes \mathcal{A}$ are induced by the irreducible representations of the Abelian group \mathcal{A} . For the contracted spin-flavor algebra

of large N_c baryons, the Abelian group consists of the generators X_0^{ia} , which satisfy the large N_c consistency condition eq. (2.3). Because the X_0^{ia} commute, states can be chosen in which X_0^{ia} are treated as coordinates,

$$(X_0^{ia})_{\text{op}} |X_0^{ia}, \dots\rangle = X_0^{ia} |X_0^{ia}, \dots\rangle, \quad (3.1)$$

so that the state $|X_0^{ia}, \dots\rangle$ is labeled by the c -number eigenvalues X_0^{ia} of the operator $(X_0^{ia})_{\text{op}}$. The ellipsis in the state vector represents other quantum numbers which will be necessary to completely specify the state. Since the states have been chosen to diagonalize the generator $(X_0^{ia})_{\text{op}}$, the distinction between the operator and its eigenvalue X_0^{ia} can be dropped. To further simplify the notation, the numbers X_0^{ia} will be treated as a 3×3 matrix X_0 , whose rows are labeled by the spin index i and whose columns are labeled by the isospin index a .

The generators of the Abelian group \mathcal{A} do not commute with the generators of \mathcal{G} . The commutation relations

$$[J^i, X_0^{jb}] = i \epsilon_{ijk} X_0^{kb}, \quad [I^a, X_0^{jb}] = i \epsilon_{abc} X_0^{jc}, \quad (3.2)$$

imply that X_0^{ia} is an irreducible tensor which transforms as $(1, 1)$ under the $SU(2)_{\text{spin}} \otimes SU(2)_{\text{flavor}}$ group. For a finite spin \otimes flavor group transformation (g, h) , X_0^{ia} transforms as

$$\begin{aligned} U_J(g)^\dagger X_0^{ia} U_J(g) &= D_{ij}^{(1)}(g) X_0^{ja}, \\ U_I(h)^\dagger X_0^{ia} U_I(h) &= D_{ab}^{(1)}(h) X_0^{ib}, \end{aligned} \quad (3.3)$$

where $U_J(g)$ is the unitary operator corresponding to a finite spin rotation by the group element g , and $D_{ij}^{(1)}(g)$ is the rotation matrix in the spin-one irreducible representation, since X_0 is an irreducible tensor operator with spin one. The isospin transformation is defined analogously. The spin one rotation matrix $D_{ij}^{(1)}$ is the familiar rotation matrix for vectors in 3-dimensional space, and will be denoted by R . Eqs. (3.1) and (3.3) give the action of finite rotations in spin and isospin on the basis states $|X_0, \dots\rangle$,

$$\begin{aligned} U_J(g) |X_0, \dots\rangle &= |R(g)X_0, \dots'\rangle, \\ U_I(h) |X_0, \dots\rangle &= |X_0 R^{-1}(h), \dots'\rangle, \end{aligned} \quad (3.4)$$

where the matrix notation for X_0^{ia} is used. The prime on the unspecified labels \dots is a reminder that the spin and isospin transformations may affect these indices. The infinitesimal form of these relations shows that \mathbf{J} and \mathbf{I} can be represented in the $|X_0, \dots\rangle$ basis as differential operators

$$\begin{aligned} J^i &= -i \epsilon_{ijk} X_0^{jc} \frac{\partial}{\partial X_0^{kc}} + \dots, \\ I^a &= -i \epsilon_{abc} X_0^{kb} \frac{\partial}{\partial X_0^{kc}} + \dots, \end{aligned} \tag{3.5}$$

where the ellipsis denotes operators acting on the \dots part of $|X_0, \dots\rangle$.

An irreducible representation of $\mathcal{G} \ltimes \mathcal{A}$ can now be constructed. First pick a reference state $|\overline{X}_0, \dots\rangle$ in the irreducible representation. All other states in the irreducible representation containing $|\overline{X}_0, \dots\rangle$ are obtained from the reference state by applying group transformations (by the definition of an irreducible representation). The reference state is an eigenvector of the generators X_0^{ia} , so group transformations generated by the X_0^{ia} only change the phase of $|\overline{X}_0, \dots\rangle$, and do not produce additional states. Group transformations generated by \mathbf{J} and \mathbf{I} change the value of X_0 , and produce new states. Let the state $|\overline{X}_0, \dots\rangle$ represent a point \overline{X}_0 in the 9-dimensional space of 3×3 matrices, with coordinates \overline{X}_0^{ia} . The group transformation $U_J(g)$ on the state $|\overline{X}_0, \dots\rangle$ gives a new point with coordinates given by $R(g)\overline{X}_0$, and similarly for $U_I(h)$. The set of all such points as g and h vary over all possible $SU(2)$ matrices is called the orbit of \overline{X}_0 , and is the set of all points $R(g)\overline{X}_0 R^{-1}(h)$ obtained by applying arbitrary spin and isospin transformations g and h , respectively, to \overline{X}_0 . Each transformation matrix is parameterized by three Euler angles, so the orbit of \overline{X}_0 is at most a 6-dimensional subspace of the 9-dimensional space of X_0 's.

The 9-dimensional space of all possible 3×3 matrices X_0^{ia} can be divided up into different disjoint orbits. Each orbit corresponds to a different irreducible representation, since states are in the same orbit if and only if they are connected by a group transformation. The different orbits can be classified in a simple manner. Any matrix \overline{X}_0 can be brought to real diagonal form,

$$\overline{X}_0 \rightarrow \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{3.6}$$

by a transformation $\overline{X}_0 \rightarrow R(g)\overline{X}_0 R^{-1}(h)$, with at most one λ_i being negative[†] and $\lambda_1 \leq \lambda_2 \leq \lambda_3$. The reference point \overline{X}_0 on each orbit can be chosen to be a matrix in the standard form eq. (3.6). Any rescaling of the X_0 's can be reabsorbed into a redefinition of the coupling constant g of eq. (2.1), so a normalization convention for the X_0 's can be imposed,

$$X_0^{ia} X_0^{ia} = \text{Tr } X_0^2 = 3. \quad (3.7)$$

This normalization condition restricts the \overline{X}_0 's to have $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3$. Any \overline{X}_0 other than the trivial representation with all $\lambda_i = 0$ can be brought into this standard form by a trivial rescaling. The trivial representation which only contains $X_0 = 0$ is not of interest for the problem under study, since it corresponds to vanishing pion-baryon couplings. Hence, all non-trivial irreducible representations are classified by \overline{X}_0 's of the form eq. (3.6) with $\text{Tr } \overline{X}_0^2 = 3$.

The unspecified labels ... of the irreducible representation transform under the little group of X_0 . The little group \mathcal{G}_X of a point X_0 on an orbit is the set of $SU(2)_{\text{spin}} \otimes SU(2)_{\text{isospin}}$ transformations (g, h) which leave the point X_0 unchanged, so that $R(g)X_0 R^{-1}(h) = X_0$. Since the different points on an orbit are connected to each other by group transformations, the little groups at different points on the same orbit are isomorphic. For example, suppose the point X'_0 is obtained from X_0 by the group transformation (g_0, h_0) , *i.e.* $X'_0 = R(g_0)X_0 R^{-1}(h_0)$. The transformation (g', h') leaves X'_0 invariant if and only if (g, h) leaves X_0 invariant, where $g' = gg_0$ and $h' = hh_0$. Any point X_0 on the orbit of \overline{X}_0 can be obtained from \overline{X}_0 by a group transformation $(g, h) \in \mathcal{G}$. If (g, h) transforms \overline{X}_0 into X_0 , then so does $(g, h)(g', h')_l$, where $(g', h')_l$ is an element of the little group $\mathcal{G}_{\overline{X}_0}$. Thus elements on the orbit of \overline{X}_0 are in one-to-one correspondence with elements of the coset space $\mathcal{G}/\mathcal{G}_{\overline{X}_0}$.

It is easy to work out the little group at the standard configuration eq. (3.6). If all the λ_i are different, the little group consists of $Z_2 \times Z_2$ where the Z_2 's are generated by 2π rotations $U_J(2\pi)$ and $U_I(2\pi)$ in spin and isospin, respectively. If two of the λ_i 's are equal and non-zero, the little group is $U(1) \times Z_2$, where the $U(1)$ is spin-isospin rotations in the two-plane of degenerate eigenvalues with $g = h$, and the Z_2 is a 2π rotation in space. If two of the λ_i 's are equal and zero, the little group is $U(1) \times U(1)$, with independent $U(1)$ rotations in spin and isospin. If three of the eigenvalues are equal and non-zero, the little

[†] The sign of any two eigenvalues can be changed simultaneously, but the sign of a single eigenvalue cannot be changed since $\det R = 1$.

group is $SU(2) \times Z_2$, where the $SU(2)$ is spin-isospin rotations with $g = h$ and the Z_2 is a 2π rotation in space. If all three eigenvalues are zero, the little group is $SU(2) \times SU(2)$.

If the little group is non-trivial, the additional state labels \dots correspond to irreducible representations of the little group. The large N_c baryon states which will be considered extensively in this paper are irreducible representations which belong to the orbit with standard configuration

$$\overline{X}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.8)$$

and little group $SU(2) \times Z_2$. The generators of the $SU(2)$ little group of \overline{X}_0 are

$$\mathbf{K} = \mathbf{I} + \mathbf{J}, \quad (3.9)$$

and the irreducible representations are $|K, k\rangle$ with $K = 0, 1/2, 1, \dots$. The irreducible representations of Z_2 are \pm , depending on whether the state is even or odd under 2π rotations, *i.e.* whether it is bosonic or fermionic. The state $|\overline{X}_0, K, k, \pm\rangle$ transforms as an irreducible representation under the little group,

$$\begin{aligned} U_K(g) |\overline{X}_0, K, k, \pm\rangle &= |\overline{X}_0, K, k, \pm\rangle D_{k'k}^{(K)}(g), \\ U_J(2\pi) |\overline{X}_0, K, k, \pm\rangle &= \pm |\overline{X}_0, K, k, \pm\rangle, \end{aligned} \quad (3.10)$$

where $D^{(K)}$ are the $SU(2)$ rotation matrices in the $(2K+1)$ dimensional representation. Note that the value of \overline{X}_0 is unchanged by the action of the little group. The irreducible representation of the little group eq. (3.10) induces an irreducible representation of the full symmetry group when combined with the transformation of X_0 , eq. (3.3). The discrete Z_2 label separates the baryon representations into fermions and bosons. For N_c odd, baryon states are fermions and the Z_2 label is restricted to be $-$. The N_c even case corresponds to baryon states which are bosons, and Z_2 label $+$. This case is not of physical interest since $N_c = 3$ in QCD. For the remainder of this paper, N_c is understood to be odd, and the Z_2 label of the states and the Z_2 factor of the little group are omitted.

It is useful to specify a single group element in \mathcal{G} connecting each point X_0 on the orbit of \overline{X}_0 with the reference point \overline{X}_0 . For the special case of interest, the group transformations are of the form (g, h) , and the little group is the set of elements (g, g) . A convenient choice in this case is to use the group element $(1, hg^{-1})$ to represent a point

on the orbit with $X_0 = R(g)\overline{X}_0R^{-1}(h)$. (Another possible choice is to use $(gh^{-1}, 1)$ as a representative element.) The states $|X_0, K, k\rangle$ are defined by

$$|X_0, K, k\rangle = U_I(hg^{-1}) |\overline{X}_0, K, k\rangle, \quad (3.11)$$

where $X_0 = R(g)\overline{X}_0R^{-1}(h)$. The connection eq. (3.11) relates the basis states $|K, k\rangle$ at the reference point \overline{X}_0 to the basis states at point X_0 of the orbit. The connection is convention dependent, but different conventions are equivalent. A familiar example of such a problem is the motion of a spin-1/2 particle on a sphere, with the spin of the particle constrained to be tangential to the sphere. Two tangential directions, say \hat{x} and \hat{y} , can be chosen at the north pole of the sphere. The tangential directions at other points are defined by parallel transport. Different paths from the north pole to a given point P give different definitions of the tangent vectors at P , and a standard path must be chosen to define basis vectors. Although the connection is convention dependent, the tangent plane spanned by the basis vectors at P is well-defined, and independent of the choice of path. This example is analogous to the definition eq. (3.11). The state $|X_0, K, k\rangle$ depends on the choice of group transformation to go from \overline{X}_0 to X_0 , but the space of linear superpositions $\sum_k c_k |X_0, K, k\rangle$ is well-defined at each point X_0 . (A more elegant formulation of this construction in terms of fiber bundles is left to the reader.)

Eqs. (3.10) and (3.11) determine the transformation law for $|X_0, K, k\rangle$ under a general group transformation (g, h) . Let X_0 be obtained from \overline{X}_0 by the transformation $(1, g_X)$. Then

$$\begin{aligned} U_J(g)U_I(h) |X_0, K, k\rangle &= U_J(g)U_I(h)U_I(g_X) |\overline{X}_0, K, k\rangle \\ &= U_I(hg_Xg^{-1})U_I(g)U_J(g) |\overline{X}_0, K, k\rangle = U_I(hg_Xg^{-1})U_K(g) |\overline{X}_0, K, k\rangle \\ &= U_I(hg_Xg^{-1}) |\overline{X}_0, K, k'\rangle D_{k'k}^{(K)}(g) = |\overline{X}_0R^{-1}(hg_Xg^{-1}), K, k'\rangle D_{k'k}^{(K)}(g) \\ &= |R(g)X_0R^{-1}(h), K, k'\rangle D_{k'k}^{(K)}(g). \end{aligned} \quad (3.12)$$

Eq. (3.12) generalizes the transformation law eq. (3.4) to include the transformation of the $|K, k\rangle$ part of the state. The generalization of eq. (3.5) is obtained by taking the infinitesimal form of eq. (3.12),

$$\begin{aligned} J^i &= -i \epsilon_{ij\ell} \delta_{k'k} X_0^{jc} \frac{\partial}{\partial X_0^{\ell c}} + T_{k'k}^{(K)i}, \\ I^a &= -i \epsilon_{abc} \delta_{k'k} X_0^{\ell b} \frac{\partial}{\partial X_0^{\ell c}}, \end{aligned} \quad (3.13)$$

where $T^{(K)}$ are $SU(2)$ generators in the $(2K+1)$ dimensional representation. The asymmetry between \mathbf{J} and \mathbf{I} in eq. (3.13) occurs because group elements of the form $(1, gh^{-1})$ were chosen to represent the coset. If the convention $(hg^{-1}, 1)$ had been adopted instead of $(1, gh^{-1})$, the $T^{(K)}$ matrices would have been added to \mathbf{I} instead of to \mathbf{J} . The basis states can be chosen to have a group-invariant normalization. If $(1, g)$ transforms $\overline{X}_0 \rightarrow X_g$, then the states are normalized so that

$$\langle X_h, K, k' | X_g, K, k \rangle = \delta_{k'k} \delta(gh^{-1}) \quad (3.14)$$

where $\delta(g)$ is a δ -function on the $SU(2)$ group normalized so that $\int dg \delta(g) = 1$.

A similar construction works for the other orbits, which have different little groups. The above recipe for constructing the irreducible representations of the large N_c spin-flavor symmetry group can be summarized: (i) pick an orbit (ii) find the little group of the orbit and choose an irreducible representation of the little group. The irreducible representation of the little group induces an irreducible representation of the full group. The importance of this construction is due to a theorem of Mackey, which states that the above procedure yields all the irreducible representations of the semidirect product group $\mathcal{G} \ltimes \mathcal{A}$ [19].[†]

The induced representations constructed above are the different solutions to the consistency equations for pion-baryon scattering. The solutions are given in terms of basis states $|X_0, K, k\rangle$ which diagonalize the axial currents. It is more convenient to work with states of definite spin and isospin, *i.e.* in a basis in which \mathbf{J} and \mathbf{I} are diagonal, since baryon states have definite spin and isospin. Let X_g denote the value of X_0 obtained by acting on \overline{X}_0 with the group transformation $(1, g)$. States of definite isospin are obtained by taking linear superpositions of $|X_g, K, k\rangle$,

$$|I I_3 m; K k\rangle = \int dg D_{I_3 m}^{(I)}(g)^* |X_g, K, k\rangle. \quad (3.15)$$

Since the representation matrices $D_{I_3 m}^{(I)}(g)$ form a complete set of basis functions on the group, the states $|I I_3 m; K k\rangle$ are complete. Under an isospin rotation, $|I I_3 m; K k\rangle$ transforms as

$$\begin{aligned} U_I(h) |I I_3 m; K k\rangle &= \int dg D_{I_3 m}^{(I)}(g)^* U_I(h) |X_g, K, k\rangle \\ &= \int dg D_{I_3 m}^{(I)}(g)^* |X_{hg}, K, k\rangle = \int dg D_{I_3 m}^{(I)}(h^{-1}g)^* |X_g, K, k\rangle \\ &= \int dg D_{I_3 I_3'}^{(I)}(h^{-1})^* D_{I_3' m}^{(I)}(g)^* |X_g, K, k\rangle = |I I_3' m; K k\rangle D_{I_3' I_3}^{(I)}(h), \end{aligned} \quad (3.16)$$

[†] The proof of Mackey's theorem depends on a technical assumption that one can find a Borel set which contains exactly one point in each disjoint orbit. This assumption is valid for the large N_c spin-flavor group.

where the second line follows from eq. (3.12) and the invariance of the group measure. Eq. (3.16) implies that $|I I_3 m; K k\rangle$ is a state which transforms under isospin as $|I I_3\rangle$. A similar calculation can be done for a spin transformation,

$$\begin{aligned}
U_J(h) |I I_3 m; K k\rangle &= \int dg D_{I_3 m}^{(I)}(g)^* U_J(h) |X_g, K, k\rangle \\
&= \int dg D_{I_3 m}^{(I)}(g)^* |X_{gh^{-1}}, K, k'\rangle D_{k'k}^{(K)}(h) \\
&= \int dg D_{I_3 m}^{(I)}(gh)^* |X_g, K, k'\rangle D_{k'k}^{(K)}(h) \\
&= \int dg D_{I_3 m'}^{(I)}(g)^* D_{m'm}^{(I)}(h)^* |X_g, K, k'\rangle D_{k'k}^{(K)}(h) \\
&= |I I_3 m'; K k'\rangle D_{m'm}^{(I)}(h)^* D_{k'k}^{(K)}(h).
\end{aligned} \tag{3.17}$$

Eq. (3.17) implies that $|I I_3 m; K k\rangle$ transforms under rotations like the product of state $|K k\rangle$ and the complex conjugate of the state $|I m\rangle$. States which transform under spin rotations like $|J J_3\rangle$ are obtained by combining the k and m indices using Clebsch-Gordan coefficients,

$$\begin{aligned}
|I I_3, J J_3; K\rangle &= \sqrt{\frac{\dim I \dim J}{\dim K}} \left(\begin{array}{cc|c} J & I & K \\ J_3 & m & k \end{array} \right) |I I_3 m; K k\rangle \\
&= \sqrt{\frac{\dim I \dim J}{\dim K}} \left(\begin{array}{cc|c} J & I & K \\ J_3 & m & k \end{array} \right) \int dg D_{I_3 m}^{(I)}(g)^* |X_g, K, k\rangle,
\end{aligned} \tag{3.18}$$

with an implied sum over m and k . The normalization factor has been chosen so that the states (3.18) are normalized to unity, using the normalization eq. (3.14) for the basis states $|X_g, K, k\rangle$.

Baryon representations can be identified with the irreducible representations of definite spin, isospin and K given in eq. (3.18). The baryon states with a given K contain all states of the form (J, I) , provided $J \otimes I \in K$. The induced representations with $K = 0$ consist of an infinite tower of states with $(J, I) = (1/2, 1/2), (3/2, 3/2), (5/2, 5/2), \dots, (N_c/2, N_c/2)$. The induced representations with $K = 1/2$ correspond to an infinite tower of states $(1/2, 0), (1/2, 1), (3/2, 1), (3/2, 2), (5/2, 2), \dots$; the induced representations with $K = 1$ contain the states $(1/2, 1/2), (3/2, 1/2), (1/2, 3/2), (3/2, 3/2), (5/2, 3/2), (3/2, 5/2), (5/2, 5/2), \dots$; and the induced representations with $K = 3/2$ include the states $(3/2, 0), (1/2, 1), (3/2, 1), (5/2, 1), \dots$. Pions with p -wave couplings to baryons carry the (J, I) quantum numbers $(1, 1)$ and $K = 0$. Thus, pions only connect baryons within a given K sector.

The quantum numbers of the baryon states in the K sectors can be identified with the known baryon spin-1/2 octet and spin-3/2 decuplet states of QCD if the quantum number K labels baryon sectors with differing strangeness. The $K = 0$ sector contains strangeness zero baryons such as the nucleon, which is identified with the state $(1/2, 1/2)$, and the Δ , which corresponds to the state $(3/2, 3/2)$. The $K = 1/2$ sector contains the strangeness -1 baryons: the $\Lambda(1/2, 0)$, $\Sigma(1/2, 1)$ and $\Sigma^*(3/2, 1)$. The $K = 1$ sector contains strangeness -2 baryons: the $\Xi(1/2, 1/2)$ and $\Xi^*(3/2, 1/2)$; and the $K = 3/2$ sector contains strangeness -3 baryons such as the $\Omega^-(3/2, 0)$. The other states in the towers correspond to baryons which exist for $N_c \rightarrow \infty$, but not for $N_c = 3$. The identification of baryons with different irreducible representations of the spin-flavor group is discussed in more detail in Sect. 5.

4. Skyrmions

The $SU(2)$ Skyrme model provides an explicit realization of the contracted $SU(4)$ spin-flavor algebra of Sect. 2. The induced representations of Sect. 3 are in one-to-one correspondence with soliton solutions of the Skyrme model. The $SU(2)$ Skyrme model with the spherical hedgehog solution Σ_0 corresponds to the induced representation with $K = 0$ and little group $SU(2) \times Z_2$. The $SU(2)$ Skyrmion is a soliton of the chiral Lagrangian of the form

$$\Sigma_0(\mathbf{x}) = e^{i\boldsymbol{\tau} \cdot \hat{\mathbf{x}} F(r)} , \quad (4.1)$$

where $F(0) = -\pi$ and $F(\infty) = 0$.[†] The Skyrmion configuration Σ_0 corresponds to the reference state $|\overline{X}_0, 0, 0\rangle$, which is invariant under $\mathbf{K} = \mathbf{I} + \mathbf{J}$,

$$(\mathbf{I} + \mathbf{J}) \Sigma_0 = 0 . \quad (4.2)$$

An isospin transformation of the soliton Σ_0 gives an equivalent soliton solution with $\Sigma = A\Sigma_0 A^{-1}$. Spin and isospin transformations generate infinitesimal body and space centered rotations on the soliton Σ , with

$$A \rightarrow AU^{-1}, \quad A \rightarrow UA, \quad (4.3)$$

respectively. The axial vector current in the Skyrme model is equal to

$$A^{ia} = \frac{1}{2} N_c g_A \text{Tr} (A \tau^i A^{-1} \tau^a) \quad (4.4)$$

[†] There is a sign error in the choice of F in ref. [6].

to leading order in N_c , where the coupling constant g_A is a function of the shape function $F(r)$ of the soliton. Thus, the $K = 0$ soliton states $\Sigma = A\Sigma_0 A^{-1}$ correspond to the states $|X_0, 0, 0\rangle$ with

$$X_0^{ia} = \frac{1}{2} \text{Tr} (A \tau^i A^{-1} \tau^a). \quad (4.5)$$

The reference point of the representation, $\overline{X}_0^{ia} = \delta^{ia}$, corresponds to the standard soliton configuration Σ_0 with $A = 1$. The collective coordinate A of the soliton determines the coordinate X_0^{ia} by eq. (4.5). The commutation relation (2.3) is satisfied in the Skyrme model because the collective coordinate A is a c -number.

Spherical hedgehog solutions with $K \neq 0$ correspond to the induced representations $|X_0, K, k\rangle$. Skyrme model solutions with $K \neq 0$ are constructed below. The approach which is adopted is closely related to a method introduced by Callan and Klebanov [20]. Callan and Klebanov showed that Skyrmions containing a single strange quark can be treated as bound states of an $SU(2)$ Skyrmion and a K meson. This method allows strange baryons to be studied in the Skyrme model without assuming $SU(3)$ symmetry. The Skyrme representation of baryons containing strange quarks given below considers bound states of Skyrmions and s -quarks.

First consider a baryon containing a single strange quark as a bound state of an $SU(2)$ soliton and a K meson. The $SU(2)$ soliton, which is a fermion, can be combined with the \overline{u} or \overline{d} antiquark in the K meson to produce a bosonic $SU(2)$ soliton, which is a color $\overline{3}$. The strange quark baryon is the colorless bound state of this bosonic soliton and a strange quark. Because the color indices of the bosonic soliton and s -quark must be contracted, it is possible to instead regard the strange quark as a colorless bosonic object with spin-1/2 and the bosonic soliton as a color singlet state. Baryons with N_s strange quarks arise as bound states of a fermionic or bosonic soliton and N_s strange quarks,

$$|\Sigma_0\rangle |sss \dots s\rangle, \quad (4.6)$$

where each strange quark carries spin 1/2. The soliton-quark bound state is completely symmetric under the exchange of the s -quarks, so $|sss \dots s\rangle$ has the strange quark spins combined into total spin $N_s/2$. The induced representation $|X_0, K, k\rangle$ can be identified with a Skyrmion bound to $2K$ strange quarks,

$$|X_0, K, k\rangle \leftrightarrow |A\Sigma_0 A^{-1}\rangle |sss \dots s\rangle, \quad (4.7)$$

where X_0 is related to A by eq. (4.4), and the spins of the $2K$ s -quarks are combined into a state with spin K and spin eigenvalue k . The operators

$$\begin{aligned} J_{ud}^i &= -i \epsilon_{ij\ell} \delta_{k'k} X_0^{jc} \frac{\partial}{\partial X_0^{\ell c}}, \\ I^a &= -i \epsilon_{abc} \delta_{k'k} X_0^{\ell b} \frac{\partial}{\partial X_0^{\ell c}}, \end{aligned} \tag{4.8}$$

generate infinitesimal space and body centered rotations of the soliton, and can be interpreted as the spin of the light degrees of freedom (u and d quarks, gluons, orbital angular momentum, *etc.*), and the isospin, respectively. The operator

$$J_s^i = T_{k'k}^{(K)i}, \tag{4.9}$$

is the strange quark spin, and acts only on the strange quarks. The total angular momentum $J = J_{ud} + J_s$ reproduces eq. (3.13).

Quantization of non-spherical soliton solutions of the chiral Lagrangian yields the induced representations of Sect. 3 with general reference points eq. (3.6). These solutions are unimportant for the study of the lowest-lying baryons because a non-spherical soliton differs in mass from the spherical soliton by an amount of order N_c .

5. Large N_c Counting Rules

The $1/N_c$ expansion for baryons relies heavily on large N_c power counting rules for baryon scattering processes and matrix elements. In this section, the large N_c behavior of f_π , M and g_A are presented. In addition, the identification of the lowest lying baryon states with the induced representations of Sect. 3 is discussed from a quark model approach. Some of the results in this section are well known, and can be found in refs. [1], [2] and [21].

5.1. Meson Green Functions

First consider the large N_c dependence of meson Green functions. The pion is created from the vacuum by a color singlet axial vector quark bilinear,

$$A^{\mu a} = \sum_{\alpha=1}^{N_c} \bar{q}^\alpha \gamma^\mu \gamma_5 \tau^a q_\alpha, \tag{5.1}$$

where τ^a is a flavor matrix, and the sum on colors is shown explicitly. The two-point function $\langle 0 | A^{\mu a} A^{\nu b} | 0 \rangle$ is dominated in the large N_c limit by planar graphs bounded by a single quark line, as shown in fig. 2, and is order N_c . The axial current two-point function is related to the pion decay constant

$$N_c \sim \int d^4x e^{ip \cdot x} \langle 0 | A^{\mu a}(x) A^{\nu b}(0) | 0 \rangle = -\frac{f_\pi^2 p^\mu p^\nu}{p^2} + \dots, \quad (5.2)$$

where the ellipsis denotes terms other than the single pion pole. The omitted terms cannot cancel the pion pole, so $f_\pi^2 \sim N_c$. In other words, $A^{\mu a} / \sqrt{N_c}$ creates pions from the vacuum with an amplitude that is finite as $N_c \rightarrow \infty$.

The above argument does not depend on the special form of the axial current; any two-point function of quark bilinears is order N_c , so any quark bilinear creates a meson with amplitude $\sqrt{N_c}$. This fact immediately implies that multi-meson vertices are suppressed by powers of $\sqrt{N_c}$. For example, an n -meson amplitude is obtained by studying the n -point function of quark bilinears. The dominant graphs are of the form shown in fig. 3, and are proportional to N_c . Each quark bilinear produces a meson with amplitude $\sqrt{N_c}$, so that the n -meson amplitude is of order $N_c / (\sqrt{N_c})^n \sim N_c^{1-n/2}$. Each additional meson produces a suppression factor of $1/\sqrt{N_c}$ in a multi-meson amplitude.

5.2. Baryon Green Functions

Large N_c baryons are color singlet states containing N_c quarks with color indices contracted using the N_c -index ϵ -symbol of $SU(N_c)$. The matrix element of a quark bilinear such as the axial current $A^{\mu a}$ between baryon states is given by the graphs in fig. 4, where the operator can be inserted on any of the N_c quark lines. Each of the insertions gives a contribution of order one, so the net contribution from the N_c diagrams is at most of order N_c . Note that the contribution is *at most* of order N_c , since there may be cancellations amongst the N_c diagrams.

We assume that the lowest lying baryon states are completely symmetric in the spatial coordinates of the quarks, and therefore must be completely symmetric in $\text{spin} \otimes \text{flavor}$, since they are totally antisymmetric in color and the quarks are fermions. It is useful to replace the quark fields by equivalent color singlet spin-1/2 boson fields which carry the spin and flavor indices of the original quarks. This convention is merely a notational convenience to obtain the correct spin-flavor quantum numbers of the baryons while avoiding color indices on the quark fields. The baryon annihilation operator is $q_{\iota_1 \alpha_1} q_{\iota_2 \alpha_2} \dots q_{\iota_{N_c} \alpha_{N_c}}$ in this new notation, where ι_i and α_i are the spin and flavor indices of the i^{th} quark, respectively.

The N_c dependence of certain baryon Green functions can be obtained by using a trick. Assume that the baryon mass is non-zero in the large N_c limit, so that one can go to the baryon rest frame. In the baryon rest frame, one can split the quark spinor field into its upper two and lower two components denoted by superscripts (\pm) , respectively, using the projector $(1 \pm \gamma^0)/2$. Consider $q_{\iota\alpha}^{(+)}$ where the spinor index ι now takes on the values 1, 2 corresponding to the two upper components of the spinor field. (Note that there is no assumption that the quark q is non-relativistic.) Define the baryon annihilation operator

$$B_\Omega = q_{\iota_1\alpha_1}^{(+)} q_{\iota_2\alpha_2}^{(+)} \dots q_{\iota_{N_c}\alpha_{N_c}}^{(+)} \Omega_{\iota_1\alpha_1} \Omega_{\iota_2\alpha_2} \dots \Omega_{\iota_{N_c}\alpha_{N_c}}, \quad (5.3)$$

where $\Omega_{\iota\alpha}$ is an arbitrary 2×2 matrix with one spin index ι in the spin-1/2 representation and one flavor index α in the isospin-1/2 representation. Baryons containing strange quarks will be discussed later in this section. The baryon state annihilated by this operator will be denoted by $|\Omega\rangle$. In the rest frame of the baryon, one has $SU(2)$ rotational symmetry and $SU(2)$ flavor symmetry. (The Lorentz boost symmetry has been broken by the choice of a particular reference frame.) Ω transforms as

$$\Omega_{\iota_1\alpha_1} \rightarrow D_{\iota_1\iota_2}^{(1/2)}(g) D_{\alpha_1\alpha_2}^{(1/2)}(h) \Omega_{\iota_2\alpha_2}, \quad (5.4)$$

under arbitrary spin and isospin transformations g and h , respectively. In the large N_c limit, the state $|\Omega\rangle$ is orthogonal to $|\Omega'\rangle$ if $\Omega \neq \Omega'$. The overlap $\langle\Omega|\Omega'\rangle$ involves the overlap of the quark $q_{\iota\alpha}\Omega_{\iota\alpha}$ with the quark $q_{\iota\alpha}\Omega'_{\iota\alpha}$ to the N_c^{th} power. The quark overlap is less than one if the two quarks are in different (not necessarily orthogonal) states, and so the baryon overlap vanishes in the limit $N_c \rightarrow \infty$.

It is instructive to consider a simpler example which illustrates the utility of the $|\Omega\rangle$ states. Consider the case of a single heavy quark flavor, so that the quark can be treated as non-relativistic. One can then define the N_c -quark state $|\hat{n}\rangle$ as the state annihilated by $q_{\hat{n}}q_{\hat{n}} \dots q_{\hat{n}}$, where $q_{\hat{n}}$ annihilates a quark with spin up along the \hat{n} direction. The overlap of a spin-1/2 state with spin up along \hat{n} and a spin-1/2 state with spin up along \hat{n}' is $\cos\theta/2$ (up to a phase), where θ is the angle between \hat{n} and \hat{n}' . Thus the overlap

$$|\langle\hat{n}|\hat{n}'\rangle| = |\cos\theta/2|^{N_c} \xrightarrow{N_c \rightarrow \infty} 0 \quad \text{if } \theta \neq 0. \quad (5.5)$$

The state $|\hat{z}\rangle$ is the state $|\uparrow\uparrow \dots \uparrow\rangle$ with $J = N_c/2$ and $J_3 = N_c/2$, and is the highest weight state. All other states with $J = N_c/2$ can be obtained from $|\hat{z}\rangle$ by applying spin lowering operators. One could equally well have started from the state $|\hat{n}\rangle$, which corresponds to

picking the state with the largest value of J_3 along the \hat{n} axis, and obtained all other states by applying lowering operators along the \hat{n} axis. Similarly, the state $|\Omega\rangle$ defined above can be thought of as a spin-flavor highest weight state, from which the other spin and flavor states are obtained by applying spin and flavor lowering operators.

The matrix element of the Hamiltonian between highest weight states $|\Omega\rangle$ is given by inserting the Hamiltonian on any one of the quark lines. There are N_c graphs which add constructively since all N_c quarks are identical. Thus, $\langle\Omega|\mathcal{H}|\Omega\rangle$ is of order N_c . Any operator that acts on a finite number of quarks (such as the Hamiltonian which is a single quark operator) cannot affect the orthogonality of $|\Omega\rangle$ for different values of Ω , so

$$\langle\Omega'|\mathcal{H}|\Omega\rangle = \begin{cases} N_c f(\Omega), & \Omega = \Omega', \\ 0, & \Omega \neq \Omega', \end{cases} \quad (5.6)$$

where $f(\Omega)$ is a function of Ω which is a spin and flavor singlet, *i.e.* it is invariant under the transformation eq. (5.4). An example of such a function is $f(\Omega) = f_0 + f_1 \epsilon^{\iota_1 \iota_2} \epsilon^{\alpha_1 \alpha_2} \Omega_{\iota_1 \alpha_1} \Omega_{\iota_2 \alpha_2}$, where f_0 and f_1 are constants. Although the above argument shows that matrix elements of the Hamiltonian \mathcal{H} between states $|\Omega\rangle$ is of order N_c , it does not imply that the states are degenerate since the order N_c term in the mass can be a function of Ω , and there can also be terms of order one in the mass.

A similar argument can be used to show that axial current matrix elements between Ω states are of order N_c . The axial current operator can be inserted on any one of the N_c identical quark lines. There is no cancellation between the different possible insertions, since all the quarks are identical. The quark axial current operator is a single quark operator, which cannot affect the orthogonality relation for the $|\Omega\rangle$'s, so

$$\langle\Omega'|A^{ia}|\Omega\rangle = \begin{cases} N_c f^{ia}(\Omega), & \Omega = \Omega', \\ 0, & \Omega \neq \Omega', \end{cases} \quad (5.7)$$

where $f^{ia}(\Omega)$ is a function of Ω which transforms like a tensor with spin one and isospin one under the transformation eq. (5.4). One such function is $f^{ia}(\Omega) = \Omega_{\iota_1 \alpha_1} \Omega_{\iota_2 \alpha_2} (\sigma^i)^{\iota_1 \iota_2} (\tau^a)^{\alpha_1 \alpha_2}$. Thus, the matrix elements of the axial current between $|\Omega\rangle$ states is of order N_c .

Note that it is not possible to prove that the matrix elements of the spin operator \mathbf{J} or isospin operator \mathbf{I} are of order N_c using this method, since it is not possible to construct a function with spin one and isospin zero (the quantum numbers of \mathbf{J}) or isospin one and spin zero (the quantum numbers of \mathbf{I}) from Ω , which is a c -number and transforms as $(1/2, 1/2)$ under $\text{spin} \otimes \text{flavor}$. The only tensors that can be constructed from the m^{th} power of Ω are in the totally symmetric tensor product $(1/2, 1/2)^{\otimes m}$, which does not contain $(1, 0)$ or $(0, 1)$.

5.3. The Relation of $|\Omega\rangle$ States to Induced Representations

Baryons annihilated by B_Ω are completely symmetric in $\text{spin} \otimes \text{flavor}$. The baryons have the quantum numbers of states in the completely symmetric tensor product $(1/2, 1/2)^{\otimes N_c}$, which are states with (J, I) equal to $(1/2, 1/2), (3/2, 3/2), \dots, (N_c/2, N_c/2)$. In the limit that $N_c \rightarrow \infty$, it is easy to see that they correspond to the induced representation discussed in the previous section with $K = 0$, and $\pm = +$ if the $N_c \rightarrow \infty$ limit is taken with N_c even, and with $\pm = -$ if the limit is taken with N_c odd. The connection between the states $|\Omega\rangle$ defined in this section and the induced representation defined in the previous section is

$$X_0^{ia} \leftrightarrow \frac{1}{2} \text{Tr} \Omega^{-1} \tau^i \Omega \tau^a, \quad (5.8)$$

where τ are the Pauli matrices for $SU(2)$.[†] Note that the point $\Omega = 1$ corresponds to the reference point \overline{X}_0 of the orbit. The quark picture naturally gives orbits where \overline{X}_0 has all three eigenvalues equal since the quark representation labels states by Ω , which transforms as $(1/2, 1/2)$ under $\text{spin} \otimes \text{isospin}$. The only object which can be constructed out of Ω that transforms as $(1, 1)$ like X_0 is a bilinear in Ω of the form eq. (5.8) which has all eigenvalues equal.

A similar analysis can be done for baryons containing a finite number of strange quarks as $N_c \rightarrow \infty$. The baryon annihilation operator

$$q_{l_1 \alpha_1}^{(+)} q_{l_2 \alpha_2}^{(+)} \dots q_{l_{N_c - N_s} \alpha_{N_c - N_s}}^{(+)} \Omega_{l_1 \alpha_1} \Omega_{l_2 \alpha_2} \dots \Omega_{l_{N_c - N_s} \alpha_{N_c - N_s}} s_{k_1}^{(+)} s_{k_2}^{(+)} \dots s_{k_{N_s}}^{(+)}, \quad (5.9)$$

annihilates baryons containing N_s strange quarks. The annihilation operator is completely symmetric under the exchange of the strange quark indices, so the spins of the N_s strange quarks are combined to form a completely symmetric state of spin $N_s/2$. These states correspond to the induced representation discussed in the previous section with $K = N_s/2$.

The results of this section strongly suggest that the lowest lying baryons correspond to induced representations on orbits with \overline{X}_0^{ia} proportional to δ^{ia} , and with $2K$ equal to the number of strange quarks in the baryon. However, this identification is not rigorous because it has not been proved that the baryons annihilated by the operators eqs. (5.3) and (5.9) are the lowest lying baryons. For example, the operator eq. (5.3) with an additional derivative on one of the $q^{(+)}$'s, or with a $q^{(+)}$ replaced by a $q^{(-)}$ transforms under a different representation of the spin-flavor algebra. One expects that these states correspond to excited baryons, but this identification has not been proved.

[†] To be precise, the Ω in eq. (5.8) transforms as a $(\overline{2}, 2)$ and is obtained from the Ω in eq. (5.3) which transforms as a $(2, 2)$ by raising one index using the $SU(2)$ epsilon symbol.

6. Pion-Baryon Couplings

The irreducible representations for the baryons constructed in Sect. 3 give the possible solutions to the pion-baryon consistency conditions. The solutions can be classified by the quantum number K , where $2K$ is equal to the number of strange quarks in the baryons. Thus, the $K = 0$ sector contains the strangeness zero baryons (N, Δ); the $K = 1/2$ sector contains the strangeness -1 baryons ($\Lambda, \Sigma, \Sigma^*$); the $K = 1$ sector contains the $S = -2$ baryons (Ξ, Ξ^*); and the $K = 3/2$ sector contains the $S = -3$ baryon Ω . Pion interactions only couple baryons within a given K sector, since pions do not carry strangeness. The results of Sect. 3 allow us to determine the pion couplings within each strangeness sector in terms of an overall coupling constant $g(K)$ which can depend on K . The pion couplings for each K sector are first derived at leading order in $1/N_c$. The analysis is then extended to include $1/N_c$ corrections. It is important to remember that while the results in this section are for baryons containing strange quarks, $SU(3)$ symmetry is not assumed and the formulæ are true irrespective of the size of the strange quark mass. A different approach which uses $SU(3)$ flavor symmetry is given in Sect. 9.

The pion-baryon couplings within a given K tower can be computed using the explicit formula eq. (3.18) for the baryon states $|I I_3, J J_3; K\rangle$. The pion couplings are the matrix elements of X_0^{ia} , and are given by

$$\begin{aligned} \langle I' I'_3, J' J'_3; K | X_0^{ia} | I I_3, J J_3; K \rangle = \\ \sqrt{\frac{\dim I \dim J \dim I' \dim J'}{(\dim K)^2}} \begin{pmatrix} J' & I' & | & K \\ J'_3 & m' & | & k' \end{pmatrix} \begin{pmatrix} J & I & | & K \\ J_3 & m & | & k \end{pmatrix} \\ \int dg \int dh D_{I'_3 m'}^{(I')}(g) D_{I_3 m}^{(I)}(h)^* \langle \bar{X}_0, K, k' | U_I(g)^\dagger X_0^{ia} U_I(h) | \bar{X}_0, K, k \rangle. \end{aligned} \quad (6.1)$$

The orthogonality of the states eq. (3.14) implies that the integrals over g and h in eq. (6.1) collapse to a single integral over g . Using eq. (3.3) to evaluate $U_I(g)^\dagger X_0^{ia} U_I(g)$ gives

$$\begin{aligned} \langle I' I'_3, J' J'_3; K | X_0^{ia} | I I_3, J J_3; K \rangle = \\ \sqrt{\frac{\dim I \dim J \dim I' \dim J'}{(\dim K)^2}} \begin{pmatrix} J' & I' & | & K \\ J'_3 & m' & | & k' \end{pmatrix} \begin{pmatrix} J & I & | & K \\ J_3 & m & | & k \end{pmatrix} \\ \int dg D_{I'_3 m'}^{(I')}(g) D_{I_3 m}^{(I)}(g)^* D_{ab}^{(1)}(g)^* \bar{X}_0^{ib}, \end{aligned} \quad (6.2)$$

where $D^{(1)*} = D^{(1)}$ since the representation is real, and $\overline{X}_0^{ib} = \delta^{ib}$ by the definition of the reference point of the orbit eq. (3.8). The integral over g can be done using the identity on D matrices,

$$\int dg D_{I'_3 m'}^{(I')}(g) D_{I_3 m}^{(I)}(g)^* D_{ai}^{(1)}(g)^* = \frac{1}{\dim I'} \left(\begin{array}{cc|c} I & 1 & I' \\ I_3 & a & I'_3 \end{array} \right) \left(\begin{array}{cc|c} I & 1 & I' \\ m & i & m' \end{array} \right). \quad (6.3)$$

Substituting eq. (6.3) into eq. (6.2), and rewriting three of the Clebsch-Gordan coefficients in terms of a $6j$ -symbol times a Clebsch-Gordan coefficient gives the result

$$\begin{aligned} \langle I' I'_3, J' J'_3; K | X_0^{ia} | I I_3, J J_3; K \rangle &= (-1)^{2J'+J-I'-K} \\ &\sqrt{\dim I \dim J} \left\{ \begin{array}{ccc} 1 & I & I' \\ K & J' & J \end{array} \right\} \left(\begin{array}{cc|c} I & 1 & I' \\ I_3 & a & I'_3 \end{array} \right) \left(\begin{array}{cc|c} J & 1 & J' \\ J_3 & i & J'_3 \end{array} \right). \end{aligned} \quad (6.4)$$

For the special case $K = 0$, eq. (6.4) reduces to

$$\langle I' I'_3, J' J'_3; 0 | X_0^{ia} | I I_3, J J_3; 0 \rangle = \sqrt{\frac{\dim J}{\dim J'}} \left(\begin{array}{cc|c} I & 1 & I' \\ I_3 & a & I'_3 \end{array} \right) \left(\begin{array}{cc|c} J & 1 & J' \\ J_3 & i & J'_3 \end{array} \right), \quad (6.5)$$

which are the couplings of the $I = J$ tower containing the nucleon and Δ baryons found in ref. [4] by explicit construction. Eq. (6.4) determines the pion couplings in a given K sector up to an overall undetermined coupling constant $g(K)$. The coupling constants in the different K sectors are not related by pion-scattering, since pions have zero strangeness and do not connect the different strangeness sectors to each other. Thus the final expression for the axial current matrix elements in the large N_c limit is

$$\begin{aligned} \langle I' I'_3, J' J'_3; K | A^{ia} | I I_3, J J_3; K \rangle &= N_c g(K) (-1)^{2J'+J-I'-K} \\ &\sqrt{\dim I \dim J} \left\{ \begin{array}{ccc} 1 & I & I' \\ K & J' & J \end{array} \right\} \left(\begin{array}{cc|c} I & 1 & I' \\ I_3 & a & I'_3 \end{array} \right) \left(\begin{array}{cc|c} J & 1 & J' \\ J_3 & i & J'_3 \end{array} \right), \end{aligned} \quad (6.6)$$

where $g(K)$ is an unknown K dependent normalization.

Eqs. (6.4) and (6.6) are the $N_c \rightarrow \infty$ predictions for the pion couplings of each K tower. The $1/N_c$ corrections to the pion couplings in each K tower can also be computed by generalizing the method employed in ref. [4] to calculate $1/N_c$ corrections for the $K = 0$ tower. In order to discuss $1/N_c$ corrections to the pion couplings, it is useful to define an expansion for the operator X^{ia} ,

$$X^{ia} = X_0^{ia} + \frac{1}{N_c} X_1^{ia} + \frac{1}{N_c^2} X_2^{ia} + \dots, \quad (6.7)$$

where X_0^{ia} is the operator which describes the pion-baryon couplings at leading order, and X_n^{ia} are operators which arise as $1/N_c^n$ corrections to the large N_c limit. With this definition, the consistency condition for pion-baryon scattering becomes

$$[X^{jb}, X^{ia}] \lesssim \mathcal{O}\left(\frac{1}{N_c}\right), \quad (6.8)$$

since each pion-baryon vertex is order $\sqrt{N_c}$ and the scattering amplitude is $\mathcal{O}(1)$. Eq. (6.8) implies the commutation relation $[X_0^{ia}, X_0^{jb}] = 0$ which was used to obtain the leading order pion couplings.

A consistency condition for the $1/N_c$ correction to the axial current is obtained by considering the three-pion scattering process $\pi^a(\omega_1 + \omega_2) + B \rightarrow \pi^b(\omega_1) + \pi^c(\omega_2) + B'$ shown in fig. 5, where the incident pion has energy $\omega_1 + \omega_2$, and the outgoing pions have energies ω_1 and ω_2 , respectively. Each pion-baryon vertex is of order $\sqrt{N_c}$, so the scattering amplitude is order $N_c^{3/2}$. The sum of the Feynman graphs in fig. 5 gives a total amplitude proportional to

$$N_c^{3/2} \frac{1}{\omega_1 \omega_2 (\omega_1 + \omega_2)} (\omega_1 [X^{kc}, [X^{jb}, X^{ia}]] + \omega_2 [X^{jb}, [X^{kc}, X^{ia}]]) . \quad (6.9)$$

The large N_c power counting rules imply that the scattering amplitude (6.9) is at most of order $1/\sqrt{N_c}$ in the large N_c limit, which means that the double commutator of three X 's must vanish at least as fast as $1/N_c^2$,

$$[X^{kc}, [X^{jb}, X^{ia}]] \lesssim \mathcal{O}\left(\frac{1}{N_c^2}\right). \quad (6.10)$$

At this order, the $1/N_c$ mass splittings of the baryons also contribute to the scattering amplitude. These terms have a different energy dependence from eq. (6.9), however, and do not affect the double commutator condition eq. (6.10). They instead yield consistency conditions for the baryon mass splittings, which are presented in Sect. 10. The constraint eq. (6.10) restricts the form of the $1/N_c$ correction to the pion couplings. Eq. (6.10) implies that X_1^{ia} must satisfy

$$[X_0^{kc}, [X_0^{jb}, X_1^{ia}]] + [X_0^{kc}, [X_1^{jb}, X_0^{ia}]] = 0 . \quad (6.11)$$

Finding the complete set of solutions to eq. (6.11) is simplest in the $|X, K, k\rangle$ basis rather than in a spin-isospin eigenstate basis of baryon states. In the $|X, K, k\rangle$ basis, the X_1^{ia} are written as functions of X_0 , partial derivatives $\partial/\partial X_0$, and operators O_K acting on the k

indices. There are no operators O_K for the $K = 0$ sector (strangeness zero baryons), so X_1 is only a function of X_0 and $\partial/\partial X_0$, *i.e.* a function of X_0^{ia} , J^i and I^a . The commutator of X_0 with a polynomial in X_0 , J and I reduces the degree of the polynomial in J and I by one, since X_0 satisfies the commutation relations (2.4). Thus, the constraint eq. (6.11) implies that X_1 can be at most linear in J or I . The only solution for X_1^{ia} in the $K = 0$ tower is X_1^{ia} proportional to X_0^{ia} [4], since the two possible operators which are linear in J and I ,

$$\epsilon_{ijk} X_0^{ja} J^k, \quad \epsilon_{abc} X_0^{ib} I^c, \quad (6.12)$$

do not transform under time reversal in the same manner as an axial current. Equivalently, the operators in eq. (6.12) are commutators of J^2 and I^2 with X_0^{ia} , and can be removed by phase redefinition of the baryon states of order $1/N_c$. Because the $1/N_c$ correction to the axial currents in the $K = 0$ sector is proportional to X_0^{ia} , it can be reabsorbed into the overall normalization factor $g(K)$. Thus there are no $1/N_c$ corrections to the ratios of pion couplings in the $K = 0$ sector [4]. The Ademollo-Gatto theorem implies that conserved charges are not renormalized to first order in symmetry breaking. Large N_c QCD has a contracted spin-flavor symmetry that is broken by $1/N_c$ corrections. At first order in $1/N_c$, one can get a correction to the axial currents proportional to the lowest order values because the axial currents are not normalized by the commutation relations eq. (2.3)–(2.5). The ratios of axial couplings are determined by the contracted symmetry, and are not renormalized at first order in symmetry breaking.

To solve eq. (6.11) for $K \neq 0$ is more complicated. Define angular momenta of the light degrees of freedom and the strange quarks of the baryon by J_{ud} and J_s , respectively, where $J = J_{ud} + J_s$. The angular momentum J_{ud} acts only on the X_0 variables in $|X_0, K, k\rangle$,

$$J_{ud}^i = -i \epsilon_{ij\ell} \delta_{k'k} X_0^{jc} \frac{\partial}{\partial X_0^{\ell c}}, \quad (6.13)$$

and the angular momentum of the strange quarks acts only on the $|K, k\rangle$ variables,

$$J_s^i = T_{k'k}^{(K)i}. \quad (6.14)$$

Any irreducible tensor operator O_L with angular momentum L acting on the $|K, k\rangle$ variables is proportional to the totally symmetric and traceless tensor product of L J_s^i 's, by the Wigner-Eckart theorem. Thus any operator in a given K sector can be written as a

product of X_0^{ia} , I^a , J_{ud}^i and J_s^i , with at most one factor of J_{ud} or I and at most K factors of J_s . The possible operators can be simplified using the identities

$$\begin{aligned}
X_0^{ia} X_0^{ib} &= \delta^{ab}, \\
X_0^{ia} X_0^{ja} &= \delta^{ij}, \\
\epsilon_{ijk} X_0^{ia} X_0^{jb} &= \epsilon_{abc} X_0^{kc}, \\
X_0^{ia} I^a &= J_{ud}^i, \\
X_0^{ia} J_{ud}^i &= I^a.
\end{aligned} \tag{6.15}$$

In the $K = 0$ sector, there were no possible operators with at most one power of J_{ud} or I with the correct transformation properties to be an axial current. In the $K \neq 0$ sector, there is an operator

$$X_1^{ia} \propto J_s^i I^a \tag{6.16}$$

which satisfies eq. (6.11) since X_0 commutes with J_s , and has the correct time-reversal properties. This term can be shown to be absent by considering the process $\pi + B \rightarrow \pi + K + B'$ involving two pions and a kaon. This discussion is deferred to the next section on kaon couplings. Thus the only $1/N_c$ corrections to the axial currents are proportional to X_0 , and can be absorbed into a redefinition of $g(K)$ [4]. The pion-baryon couplings including $1/N_c$ corrections are still given by eq. (6.6). The $1/N_c$ correction to X^{ia} is proportional to X_0^{ia} , so one finds that[†]

$$[X^{ia}, X^{jb}] \lesssim \mathcal{O}\left(\frac{1}{N_c^2}\right), \tag{6.17}$$

which makes a contribution to the $\pi + B \rightarrow \pi + B'$ scattering amplitude of order $1/N_c$. The order one contribution to the scattering amplitude results from the $1/N_c$ correction to the intermediate baryon propagator due to the baryon mass splittings, which are of order $1/N_c$.

The $1/N_c^2$ correction to the axial currents is constrained by the four-pion scattering process $\pi + B \rightarrow \pi + \pi + \pi + B'$ which gives consistency conditions for the $1/N_c^2$ correction X_2^{ia} . The scattering amplitude for this process contains terms of the form $N_c^2 [X, [X, [X, X]]]$ and $N_c^2 [[X, X], [X, X]]$, which must be of order $1/N_c$, by the large N_c

[†] This commutator condition implies that the Δ contribution cancels the g_A^2 term in the Adler-Weisberger sum rule to two orders in N_c .

power counting rules. Since $[X, X]$ is of order $1/N_c^2$, the second commutator condition is automatically satisfied. The first condition places a restriction on X_2 ,

$$\left[X_0^{\ell d}, \left[X_0^{kc}, \left[X_0^{jb}, X_2^{ia} \right] \right] \right] + \left[X_0^{\ell d}, \left[X_0^{kc}, \left[X_2^{jb}, X_0^{ia} \right] \right] \right] = 0. \quad (6.18)$$

The most general solution of eq. (6.18) (and the analogous conditions obtained by replacing pions by kaons) is that X_2 can have the form $J^i I^a$, $J_s^i I^a$, $\{I^2, X_0^{ia}\}$, and $\{J^2, X_0^{ia}\}$, as well as terms proportional to the lowest order operator X_0 which can be reabsorbed into a redefinition of $g(K)$.

The coupling constant $g(K)$ depends on K and $1/N_c$. The coupling constant $g(K)$ for baryons in the K sector has the form

$$g(K) = c_0 + \frac{1}{N_c} c_1 K + \frac{1}{N_c^2} c_2 K^2 + \mathcal{O}\left(\frac{1}{N_c^3}\right), \quad (6.19)$$

where the K -independent coefficients c_i have expansions in powers of $1/N_c$, *i.e.* the term of order one is K independent, the term of order $1/N_c$ is at most linear in K , and the term of order $1/N_c^2$ is at most quadratic in K . This form for $g(K)$ is derived in the next section. Thus, $g(K)$ can be considered to be a polynomial in $1/N_c$ and K/N_c . It is important to remember that this formula was obtained without using $SU(3)$ symmetry. Ratios of pion couplings within a given K tower are given by eq. (6.6) up to corrections of order $1/N_c^2$, since $g(K)$ drops out in the ratio. Ratios of pion-couplings between two different towers can have $1/N_c$ corrections due to the K dependent term in $g(K)$ which is linear in K and of order $1/N_c$.

7. Kaon Couplings

The kaon couplings between baryons can be obtained by studying kaon-baryon scattering in the large N_c limit. Consistency conditions for processes involving both pions and kaons also restrict the form of the pion couplings discussed in the preceding section. The results of this section are derived without assuming $SU(3)$ symmetry, and are valid irrespective of the size of the strange quark mass. The analysis of kaon-baryon couplings is similar to the discussion of heavy quark meson-baryon couplings in ref. [5].

The pion axial current matrix elements are of order N_c in the large N_c limit, so that the pion-baryon vertex is of order $\sqrt{N_c}$. The axial vector matrix elements of a strangeness changing current between baryon states containing a finite number of strange quarks as

$N_c \rightarrow \infty$ are suppressed by a factor of $1/\sqrt{N_c}$ relative to the pion couplings. Consider, for example, the $\Delta S = 1$ transition matrix element between a baryon containing a single strange quark, and a baryon containing no strange quarks. The strangeness changing axial current must be inserted on the strange quark line. The baryon containing a strange quark is a linear superposition of states in which the n^{th} quark is the s -quark ($1 \leq n \leq N_c$), each with amplitude $1/\sqrt{N_c}$. The net amplitude is of order N_c (the possible insertions on any of the quark lines) times $1/\sqrt{N_c}$ (the amplitude that the given quark line is an s -quark), *i.e.* of order $\sqrt{N_c}$. Thus the kaon-baryon vertex is of order one, since f_K is of order $1/\sqrt{N_c}$. The constraints of the large N_c limit of QCD on the kaon couplings are less powerful than for pions, because the kaon-baryon vertex does not grow with N_c . Nevertheless, it is still possible to derive the kaon couplings to leading order in the large N_c limit.

A general kaon-baryon coupling is written as the baryon strangeness changing axial current matrix element

$$\langle B' | \bar{s} \gamma^i \gamma_5 q^\alpha | B \rangle = \sqrt{N_c} (Y^{i\alpha})_{B'B}, \quad (7.1)$$

times the derivatively coupled kaon field $\partial^i \bar{K}^\alpha / f_K$, where the index α represents the isospin flavor index of the kaon and the index i is a spin index. This p -wave kaon carries spin one and isospin $1/2$. The labels B and B' denote baryons in the K and $K + 1/2$ sectors, respectively, since the absorption of a \bar{K} adds a strange quark to the baryon. A similar equation for the matrix elements of $\bar{q}^\alpha \gamma^i \gamma_5 s$ defines the hermitian conjugate matrix $Y^{\dagger i\alpha}$ describing couplings of K 's to baryons. Since $f_K \sim \sqrt{N_c}$ in the large N_c limit, the baryon-kaon vertex is $\mathcal{O}(1)$.

Consider the scattering amplitude for $\bar{K}^\alpha(\omega, \mathbf{k}) + B \rightarrow \pi^b(\omega, \mathbf{k}') + B'$, shown in fig. 6, which turns a baryon B into a baryon B' with one additional strange quark. The scattering amplitude for this process is given by

$$\mathcal{A} = -i \frac{N_c^{3/2}}{f_\pi f_K} \frac{k^i k'^j}{\omega} [g(K) X^{jb}, Y^{i\alpha}]_{B'B}, \quad (7.2)$$

where the coupling $g(K)$ must be retained in the commutator because the amplitude depends on initial, final and intermediate baryons in different K sectors. Since $f_{\pi,K} \sim \sqrt{N_c}$, the scattering amplitude naively grows like $\sqrt{N_c}$. Large N_c power counting rules imply that the amplitude should be order $1/\sqrt{N_c}$, so the large N_c consistency condition for $\bar{K}^\alpha + B \rightarrow \pi^b + B'$ scattering is

$$[g(K) X^{jb}, Y^{i\alpha}] \lesssim \mathcal{O}\left(\frac{1}{N_c}\right). \quad (7.3)$$

As for the pion couplings X , define the $1/N_c$ expansion for the kaon couplings Y by

$$Y^{ia} = Y_0^{ia} + \frac{1}{N_c} Y_1^{ia} + \dots \quad (7.4)$$

Eq. (7.3) implies that $Y_0^{i\alpha}$ must satisfy

$$\left[g(K) X_0^{jb}, Y_0^{i\alpha} \right] = 0. \quad (7.5)$$

It is important to remember that the kaon coupling changes the value of K by $1/2$. Thus gX_0 in the two terms of the commutator eq. (7.5) refer to the pion couplings in two different towers with K values differing by $1/2$. The solution of eq. (7.5) is simple in the $|X_0, K, k\rangle$ basis. Taking the matrix element of eq. (7.5) yields

$$\langle X'_0, K + \tfrac{1}{2}, k' | \left[g(K) X_0^{jb}, Y_0^{i\alpha} \right] | X_0, K, k \rangle = 0. \quad (7.6)$$

Inserting a complete set of states and using eq. (3.1) gives

$$\left(g(K + \tfrac{1}{2}) X_0'^{jb} - g(K) X_0^{jb} \right) \langle X'_0, K + \tfrac{1}{2}, k' | Y_0^{i\alpha} | X_0, K, k \rangle = 0, \quad (7.7)$$

which implies that $g(K) = g(K + \frac{1}{2})$ if $X_0 = X'_0$ and that

$$\langle X'_0, K + \tfrac{1}{2}, k' | Y_0^{i\alpha} | X_0, K, k \rangle = 0 \quad \text{if } X_0 \neq X'_0. \quad (7.8)$$

The equality on $g(K)$ which is a consequence of the consistency condition eq. (7.5) can be rewritten as

$$g(K) = g(0) + \mathcal{O}\left(\frac{1}{N_c}\right), \quad (7.9)$$

which proves the assertion of the previous section that the order one contribution to $g(K)$ is independent of K .

The constraint eq. (7.8) on the kaon couplings implies that the operator Y_0 does not change the value of the collective coordinate X_0 . Thus, Y_0 can be written as a function of X_0 and operators acting on $|K, k\rangle$, with no derivatives with respect to X_0 . The matrix element of Y_0 between general X_0 states is related to the matrix element between states at the standard reference point $\overline{X}_0^{ia} = \delta^{ia}$ by a group transformation,

$$\langle X_g, K + \tfrac{1}{2}, k' | Y_0^{i\alpha} | X_g, K, k \rangle = D_{\alpha\beta}^{(1/2)}(g) \langle \overline{X}_0, K + \tfrac{1}{2}, k' | Y_0^{i\beta} | \overline{X}_0, K, k \rangle, \quad (7.10)$$

where X_g is obtained from \overline{X}_0 by an isospin rotation g , and $D_{\alpha\beta}^{(1/2)}$ is the rotation matrix in the spin-1/2 representation. The matrix element on the right-hand side of eq. (7.10)

is determined by considering the transformation properties of Y_0 under the little group generated by \mathbf{K} . Since Y_0 has spin one and isospin $1/2$, it transforms as a linear combination of irreducible tensor operators with $\Delta K = 1/2$ and $3/2$. These operators must be combined with the states $|K, k\rangle$ and $|K + \frac{1}{2}, k'\rangle$ in K -invariant linear combinations. The state $|K, k\rangle$ can be considered to be the completely symmetric tensor product of $2K$ strange quarks, each with spin- $1/2$. The state $|K + \frac{1}{2}, k'\rangle$ is then the completely symmetric tensor product of $(2K+1)$ strange quarks. Any transition operator between $|K, k\rangle$ and $|K + \frac{1}{2}, k'\rangle$ can be written in terms of products of creation and annihilation operators a_α^\dagger and a^α which create and annihilate a strange quark with spin α . To make a transition from K to $K + 1/2$, the operator must have one more a^\dagger than a . Any operator that transforms as $\Delta K = 1/2$ or $\Delta K = 3/2$ between K and $K + 1/2$ states is proportional to a linear combination of a_α^\dagger and $a_\alpha^\dagger a_\beta^\dagger a^\gamma$ by the Wigner-Eckart theorem. The first operator is pure $\Delta K = 1/2$, and the second is a linear combination of $\Delta K = 1/2$ and $\Delta K = 3/2$. Thus the most general form for $Y_0^{i\alpha}$ at the standard point \overline{X}_0 of the orbit is

$$Y_0^{i\alpha} = c(K) a_\lambda^\dagger (\sigma^i)_\alpha^\lambda + d(K) a_\alpha^\dagger (a^\dagger \sigma^i a), \quad (7.11)$$

where $c(K)$ and $d(K)$ are coefficients which depend on K . The $\Delta K = 1/2$ operator proportional to $c(K)$ preserves the s -quark spin symmetry of the baryons, whereas the operator proportional to $d(K)$ violates the s -quark spin symmetry.

The $\Delta K = 3/2$ operator is forbidden by a large N_c consistency condition obtained from $\overline{K}^\alpha + B \rightarrow K^\beta + B'$ scattering. Naively, the amplitude is of order one since each kaon-baryon vertex is of order one. However, large N_c quark counting rules show that the amplitude is at most of order $1/N_c$, which leads to the consistency condition

$$\left[Y_0^{j\beta}, Y_0^{i\alpha} \right] = 0. \quad (7.12)$$

There is an important subtlety when one considers kaon-baryon scattering. The quark counting rules show that $\overline{K}^\alpha + B \rightarrow K^\beta + B'$ is of order $1/N_c$, but the process $K^\alpha + B \rightarrow K^\beta + B'$ is of order one. Thus one obtains the consistency condition eq. (7.12) but the condition $\left[Y_0^{j\beta}, Y_0^{i\alpha} \right] = 0$ is *not satisfied*. The consistency condition eq. (7.12) requires that the coefficients $d(K)$ in eq. (7.11) vanish, so only the $\Delta K = 1/2$ amplitude is allowed at leading order in $1/N_c$. Thus, kaon-baryon couplings respect baryon s -quark spin symmetry to leading order in $1/N_c$, even though we have not assumed that the strange quark is heavy. Note that this result does not imply that strange quark spin symmetry is a good symmetry

of the theory. For example, there is no reason to believe that the couplings of the K^* to baryons are related to the couplings of the kaon to baryons by s -quark spin symmetry.

Eq. (7.11) can be restricted further by considering the scattering process $K^\alpha + B \rightarrow K^\beta + K^\gamma + B'$, which yields the constraint

$$[Y^{k\gamma}, [Y^{j\beta}, Y^{\dagger i\alpha}]] \lesssim \mathcal{O}\left(\frac{1}{N_c}\right). \quad (7.13)$$

Eq. (7.13) implies that $c(K) = c(0)$, a constant independent of K , so that

$$Y_0^{i\alpha} = c a_\lambda^\dagger (\sigma^i)^\lambda{}_\alpha \quad (7.14)$$

is determined up to an overall normalization constant c .

Kaon couplings for baryon states of definite spin and isospin can now be computed using eqs. (3.18), (7.10) and (7.14). The matrix element eq. (7.10) becomes

$$\begin{aligned} \langle X_g, K + \tfrac{1}{2}, k' | Y_0^{i\alpha} | X_g, K, k \rangle &= c \sqrt{\dim K} D_{\alpha\beta}^{(1/2)}(g) \\ &\quad \left(\begin{array}{cc|c} \tfrac{1}{2} & 1 & \tfrac{1}{2} \\ \beta & i & \gamma \end{array} \right) \left(\begin{array}{cc|c} \tfrac{1}{2} & K & K + \tfrac{1}{2} \\ \gamma & k & k' \end{array} \right), \end{aligned} \quad (7.15)$$

where the equation

$$\langle \bar{X}_0, K + \tfrac{1}{2}, k' | a_\beta^\dagger | \bar{X}_0, K, k \rangle = \sqrt{2(K + \tfrac{1}{2})} \left(\begin{array}{cc|c} \tfrac{1}{2} & K & K + \tfrac{1}{2} \\ \beta & k & k' \end{array} \right) \quad (7.16)$$

has been used. Using the definition of isospin and spin states eq. (3.18) yields after considerable manipulation

$$\begin{aligned} \langle I' I'_3, J' J'_3; K' | Y_0^{i\alpha} | I I_3, J J_3; K \rangle &= c \sqrt{\dim I \dim J \dim K \dim K'} \\ &\quad \left\{ \begin{array}{ccc} \tfrac{1}{2} & 1 & \tfrac{1}{2} \\ I & J & K \\ I' & J' & K' \end{array} \right\} \left(\begin{array}{cc|c} I & \tfrac{1}{2} & I' \\ I_3 & \alpha & I'_3 \end{array} \right) \left(\begin{array}{cc|c} J & 1 & J' \\ J_3 & i & J'_3 \end{array} \right), \end{aligned} \quad (7.17)$$

where the quantity in curly braces is the $9j$ symbol, and $K' = K + 1/2$. This equation determines all the kaon coupling ratios to leading order in $1/N_c$, without assuming $SU(3)$ symmetry.

One can also consider the scattering processes $\pi + B \rightarrow \pi + K + B'$, $\bar{K} + B \rightarrow K + \pi + B'$, and $\bar{K} + B \rightarrow K + K + B'$ which give the consistency conditions

$$[Y^{k\gamma}, [X^{jb}, X^{ia}]] = \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (7.18)$$

$$[X^{kc}, [Y^{j\beta}, Y^{i\alpha}]] = \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (7.19)$$

and

$$[Y^{k\gamma}, [Y^{j\beta}, Y^{i\alpha}]] = \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (7.20)$$

respectively. These consistency conditions can be used to show that the term in $g(K)$ in eq. (6.19) of order $1/N_c$ is at most linear in K and that X_1^{ia} does not contain terms of the form $J_s I$, as was stated in the previous section.

8. η Couplings

The matrix elements of the T^8 axial current between baryon states containing finitely many strange quarks as $N_c \rightarrow \infty$ is of order one, so that the η couplings are of order $1/\sqrt{N_c}$ with respect to the kaon couplings, and order $1/N_c$ with respect to the pion couplings. As for the pions and kaons, define the η couplings by the axial vector current

$$\langle B' | \bar{q} \gamma^i \gamma_5 T^8 q | B \rangle = (Z^i)_{B'B}, \quad (8.1)$$

times the derivatively coupled η field $\partial^i \eta / f_\eta$, where Z^i is a matrix with spin one and isospin zero. The η -baryon vertex is order $1/\sqrt{N_c}$. The first non-trivial constraint on Z^i comes from the process $\eta + B \rightarrow \pi + \pi + B'$, with amplitude proportional to

$$N_c^{1/2} [X^{ia}, [X^{jb}, Z^k]] . \quad (8.2)$$

Large N_c power counting rules require that the amplitude be order $1/\sqrt{N_c}$, so that

$$[X_0^{ia}, [X_0^{jb}, Z_0^k]] = 0 . \quad (8.3)$$

A similar argument using the processes $\eta + B \rightarrow \pi + K + B'$ and $\eta + B \rightarrow K + K + B'$ gives the constraints

$$[X_0^{ia}, [Y_0^{jb}, Z_0^k]] = 0 , \quad (8.4)$$

and

$$[Y_0^{ia}, [Y_0^{jb}, Z_0^k]] = 0 . \quad (8.5)$$

The solution to these equations is

$$Z_0^i = a J^i + b J_s^i \quad (8.6)$$

where a and b are constants independent of K to leading order in $1/N_c$. The ratios of the different η couplings are not completely determined even at leading order, since they depend on the unknown ratio a/b .

9. Meson-Baryon Couplings in the $SU(3)$ Limit and F/D

The previous three sections analyzed the pion, kaon and η couplings of the baryons without assuming $SU(3)$ symmetry. In this section, these couplings are studied in the limit of $SU(3)$ symmetry using tensor methods [22][23][24]. The large N_c results derived earlier allow us to compute the F/D ratio for the baryon axial currents for $N_c = 3$ to order $1/N_c$. They also constrain the form of non-analytic chiral logarithmic corrections to the axial currents.

The spin-1/2 baryons in the large N_c limit transform according to the $SU(3)$ tensor

$$\mathcal{B}_{j_1 j_2 \dots j_\nu}^i \quad (9.1)$$

with one upper and ν completely symmetric lower indices, where $N_c = 2\nu + 1$. For $N_c = 3$, eq. (9.1) reduces to the usual baryon octet tensor with one upper and one lower index. The spin-3/2 baryons transform according to the tensor

$$\mathcal{T}_{j_1 j_2 \dots j_{\nu-1}}^{i_1 i_2 i_3} \quad (9.2)$$

which is completely symmetric in its three upper and $(\nu - 1)$ lower indices. For $N_c = 3$, eq. (9.2) reduces to the baryon decuplet with three upper indices and no lower indices. Throughout this section, the spin-1/2 baryons are referred to as octet baryons, and the spin-3/2 baryons as decuplet baryons, even though these are the dimensions of the representations only for $N_c = 3$. What makes the $SU(3)$ analysis subtle is that the form of the $SU(3)$ tensor changes with N_c . In this section, these “representation effects” are eliminated in order to extrapolate the large N_c results consistently to $N_c = 3$.

The most general meson couplings to the baryon octet in the $SU(3)$ limit are given by the two possible invariants

$$\mathcal{M} \bar{\mathcal{B}}_a^{b_1 b_2 \dots b_\nu} (T^A)^a{}_c \mathcal{B}_{b_1 b_2 \dots b_\nu}^c + \mathcal{N} \bar{\mathcal{B}}_a^{c b_2 \dots b_\nu} \mathcal{B}_{d b_2 \dots b_\nu}^a (T^A)^d{}_c, \quad (9.3)$$

where T^A is the $SU(3)$ octet matrix corresponding to a meson of type A. These invariants reduce to the two invariants $\text{Tr} \bar{\mathcal{B}} T^A \mathcal{B}$ and $\text{Tr} \bar{\mathcal{B}} \mathcal{B} T^A$ for $N_c = 3$ with coefficients $(D + F)$ and $(D - F)$, respectively. Similarly, the meson-decuplet-octet coupling is given in terms of a single $SU(3)$ invariant tensor

$$\mathcal{L} \bar{\mathcal{T}}_{\alpha \mu \nu}^{b_1 b_2 \dots b_{\nu-1}} (T^A)^\mu{}_\beta \mathcal{B}_{\gamma b_1 b_2 \dots b_{\nu-1}}^\nu \epsilon^{\alpha \beta \gamma}. \quad (9.4)$$

The components of the baryon tensors \mathcal{B} and \mathcal{T} are [22][23][24]
for the proton:

$$\mathcal{B}_{33...3}^1 = 1, \quad (9.5)$$

for the neutron:

$$\mathcal{B}_{33...3}^2 = 1, \quad (9.6)$$

for the Σ^+ :

$$\mathcal{B}_{233...3}^1 = \frac{1}{\sqrt{\nu}}, \quad (9.7)$$

for the Σ^- :

$$\mathcal{B}_{133...3}^2 = -\frac{1}{\sqrt{\nu}}, \quad (9.8)$$

for the Σ^0 :

$$\mathcal{B}_{133...3}^1 = -\mathcal{B}_{233...3}^2 = -\frac{1}{\sqrt{2\nu}}, \quad (9.9)$$

for the Λ :

$$\mathcal{B}_{133...3}^1 = \mathcal{B}_{233...3}^2 = \frac{1}{\sqrt{4+2\nu}}, \quad \mathcal{B}_{33...3}^3 = -\frac{2}{\sqrt{4+2\nu}}, \quad (9.10)$$

for the Ξ^0 :

$$\mathcal{B}_{233...3}^3 = -\frac{3}{2} \mathcal{B}_{223...3}^2 = -3 \mathcal{B}_{123...3}^1 = \sqrt{\frac{3}{\nu(\nu+2)}}, \quad (9.11)$$

for the Δ^{++} :

$$\mathcal{T}_{33...3}^{111} = 1, \quad (9.12)$$

for the Σ^{*+} :

$$\frac{4}{3} \mathcal{T}_{133...3}^{111} = 4 \mathcal{T}_{233...3}^{112} = -\mathcal{T}_{333...3}^{113} = \sqrt{\frac{4}{3(\nu+3)}}, \quad (9.13)$$

for the Ξ^{*0} :

$$\begin{aligned} \frac{1}{2} \mathcal{T}_{333...3}^{133} &= -\frac{3}{4} \mathcal{T}_{133...3}^{113} = -\frac{3}{2} \mathcal{T}_{233...3}^{123} = \\ \mathcal{T}_{113...3}^{111} &= 3 \mathcal{T}_{123...3}^{112} = 3 \mathcal{T}_{223...3}^{122} = \sqrt{\frac{1}{(\nu+2)(\nu+3)}}, \end{aligned} \quad (9.14)$$

for the Ω^- :

$$\mathcal{T}_{333...3}^{333} = -2 \mathcal{T}_{133...3}^{331} = -2 \mathcal{T}_{233...3}^{332} = \sqrt{\frac{2}{3\nu-1}}. \quad (9.15)$$

The symmetry of the tensors \mathcal{B} and \mathcal{T} determines the other non-zero components. For example, the Σ^+ tensor has $\mathcal{B}_{233...3}^1 = \mathcal{B}_{323...3}^1 = \mathcal{B}_{332...3}^1 = \mathcal{B}_{333...2}^1 = 1/\sqrt{\nu}$, *etc.* The

coefficients of \mathcal{M} , \mathcal{N} , and \mathcal{L} obtained by using eqs. (9.5)–(9.15) in eqs. (9.3) and (9.4) are given in Tables 1 and 2, respectively.

Amplitude	\mathcal{M}	\mathcal{N}
$p \rightarrow n\pi^+$	1	0
$\Sigma^+ \rightarrow \Sigma^0\pi^+$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}\nu}$
$\Sigma^+ \rightarrow \Lambda\pi^+$	$\frac{\nu}{\sqrt{\nu(2\nu+4)}}$	$\frac{1}{\sqrt{\nu(2\nu+4)}}$
$p \rightarrow \Lambda K^+$	$-\frac{2}{\sqrt{2\nu+4}}$	$\frac{1}{\sqrt{2\nu+4}}$
$p \rightarrow \Sigma^0 K^+$	0	$-\frac{1}{\sqrt{2}\nu}$
$p \rightarrow p\eta$	1	-2
$\Sigma^+ \rightarrow \Sigma^+\eta$	1	$-2 + \frac{3}{\nu}$
$\Lambda \rightarrow \Lambda\eta$	$\frac{\nu-4}{\nu+2}$	$-\frac{2\nu+1}{\nu+2}$

Table 1

Amplitude	\mathcal{L}
$\Delta^{++} \rightarrow p\pi^+$	1
$\Sigma^{*+} \rightarrow \Sigma^0\pi^+$	$-(\nu+1)\sqrt{\frac{1}{6\nu(\nu+3)}}$
$\Sigma^{*+} \rightarrow \Lambda\pi^+$	$\sqrt{\frac{2(\nu+2)}{3(\nu+3)}}$

Table 2

The octet-octet and decuplet-octet meson couplings in the $SU(3)$ limit are obtained using the Clebsch-Gordan coefficients computed using tensor methods in Tables 1 and 2, and the unknown coefficients \mathcal{M} , \mathcal{N} and \mathcal{L} . The large N_c analysis of the preceding sections determines the ratios of pion couplings in a given K sector up to corrections of order $1/N_c^2$, and can be used to determine the ratio \mathcal{N}/\mathcal{M} to order one and \mathcal{L}/\mathcal{M} to order $1/N_c$. What makes the determination of \mathcal{N}/\mathcal{M} possible is that the $K = 1/2$ baryons states contain two

different isospin states in the octet, the Λ and Σ . The ratio of the $\Sigma \rightarrow \Sigma\pi$ coupling to the $\Sigma \rightarrow \Lambda\pi$ coupling is known to order $1/N_c$,

$$\frac{\Sigma^+ \rightarrow \Sigma^0 \pi^+}{\Sigma^+ \rightarrow \Lambda \pi^+} = \frac{\sqrt{6} \begin{Bmatrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}}{\sqrt{2} \begin{Bmatrix} 1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}} = 1 + \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (9.16)$$

where we have used eq. (6.6) for the pion couplings. The $SU(3)$ value for this ratio is obtained from Table 1 to be

$$\frac{\Sigma^+ \rightarrow \Sigma^0 \pi^+}{\Sigma^+ \rightarrow \Lambda \pi^+} = \sqrt{1 + \frac{2}{\nu}} \left(\frac{\nu \mathcal{M} - \mathcal{N}}{\nu \mathcal{M} + \mathcal{N}} \right). \quad (9.17)$$

Expanding eq. (9.17) in a power series in $1/N_c$, and comparing with eq. (9.16) gives the large N_c prediction

$$\frac{\mathcal{N}}{\mathcal{M}} = \frac{1}{2} + \frac{\alpha}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (9.18)$$

where we have denoted the (unknown) coefficient of the $1/N_c$ term in the ratio by α , because it will be needed later in this section. An inspection of Table 1 shows that knowing \mathcal{N}/\mathcal{M} to order one is sufficient to determine all the octet-pion couplings to order $1/N_c$, since the coefficient of \mathcal{N} is suppressed by $1/N_c$ relative to that of \mathcal{M} . The ratio of pion couplings for $\Delta^{++} \rightarrow p\pi^+$ to $p \rightarrow n\pi^+$ or the ratio of $\Sigma^{*+} \rightarrow \Sigma^0 \pi^+$ to $\Sigma^+ \rightarrow \Lambda \pi^+$ can be used to determine the ratio \mathcal{L}/\mathcal{M} to order $1/N_c$,

$$\frac{\mathcal{L}}{\mathcal{M}} = \frac{\sqrt{3}}{2} + \mathcal{O}\left(\frac{1}{N_c^2}\right). \quad (9.19)$$

(The normalization of \mathcal{L} relative to \mathcal{M} depends on how one normalizes the spin invariants for \mathcal{B} and \mathcal{T} . Eq. (9.19) is derived assuming that the spin invariants for $\mathcal{T} \rightarrow \mathcal{B}\pi$ and $\mathcal{B} \rightarrow \mathcal{B}\pi$ are normalized to be equal to their respective spin Clebsch-Gordan coefficients.)

The large N_c results obtained in the previous sections are consistent with $SU(3)$ symmetry. For example, the ratio

$$\frac{\Sigma^{*+} \rightarrow \Sigma^0 \pi^+}{\Sigma^{*+} \rightarrow \Lambda \pi^+} = \frac{\sqrt{6} \begin{Bmatrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{Bmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}}{\sqrt{2} \begin{Bmatrix} 1 & 0 & 1 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{Bmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}} = -\frac{1}{2} + \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (9.20)$$

is obtained using the large N_c result eq. (6.6), and is valid for any value of the s -quark mass. The same ratio is obtained using $SU(3)$ tensors and the Clebsch-Gordan coefficients

in Table 2. There is only a single $SU(3)$ invariant amplitude for decuplet-octet pion couplings, so the coupling \mathcal{L} drops out of the ratio to give

$$\frac{\Sigma^{*+} \rightarrow \Sigma^0 \pi^+}{\Sigma^{*+} \rightarrow \Lambda \pi^+} = -(\nu + 1) \sqrt{\frac{3}{4\nu(\nu + 2)}}, \quad (9.21)$$

which is valid in the $SU(3)$ limit, for any N_c . Expanding eq. (9.21) gives

$$\frac{\Sigma^{*+} \rightarrow \Sigma^0 \pi^+}{\Sigma^{*+} \rightarrow \Lambda \pi^+} = -\frac{1}{2} + \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (9.22)$$

since there is no $1/N_c$ term in the expansion of the Clebsch-Gordan coefficient ratio in eq. (9.21).

In Sect. 6, the pion couplings of the different K sectors were computed up to an overall coupling constant $g(K)$ which had an expansion of the form eq. (6.19). In the $SU(3)$ limit, the coefficient of the term linear in K can be determined. Consider the ratio

$$\frac{\Delta^{++} \rightarrow p \pi^+}{\Sigma^{*+} \rightarrow \Sigma^0 \pi^+} = \frac{g(0)}{g(\frac{1}{2})} \frac{\sqrt{4} \left\{ \begin{matrix} \frac{3}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{3}{2} & 0 \end{matrix} \right\} \left(\begin{matrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} \end{matrix} \right)}{\sqrt{6} \left\{ \begin{matrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{matrix} \right\} \left(\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \right)} = -\sqrt{6} \frac{g(0)}{g(\frac{1}{2})} \quad (9.23)$$

which follows from eq. (6.6). In the $SU(3)$ limit, the same ratio can be evaluated using the Clebsch-Gordan coefficients in Table 2 in terms of no unknowns since the coupling \mathcal{L} cancels in the ratio,

$$\frac{\Delta^{++} \rightarrow p \pi^+}{\Sigma^{*+} \rightarrow \Sigma^0 \pi^+} = -\sqrt{6} \left(1 + \frac{1}{N_c} \right) + \mathcal{O}\left(\frac{1}{N_c^2}\right). \quad (9.24)$$

Comparing eq. (9.23) with eq. (9.24) gives

$$\frac{g(0)}{g(\frac{1}{2})} = 1 + \frac{1}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (9.25)$$

so that

$$\frac{g(K)}{g(0)} = 1 - \frac{2K}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right). \quad (9.26)$$

in the $SU(3)$ limit.

The same result eq. (9.26) also can be obtained using only the octet couplings. The ratio

$$\frac{p \rightarrow n \pi^+}{\Sigma^+ \rightarrow \Lambda \pi^+} = -\frac{g(0)}{g(\frac{1}{2})} \frac{\sqrt{4} \left\{ \begin{matrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{matrix} \right\} \left(\begin{matrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{matrix} \right)}{\sqrt{2} \left\{ \begin{matrix} 1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix} \right\} \left(\begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \right)} = \sqrt{2} \frac{g(0)}{g(\frac{1}{2})} \quad (9.27)$$

using eq. (6.6). Evaluation of this ratio using the Clebsch-Gordan coefficients in Table 1 and eq. (9.18) for \mathcal{N}/\mathcal{M} gives eq. (9.25). Thus, all the pion-couplings can be determined consistently to order $1/N_c$ in the $SU(3)$ limit.

The F/D ratio for the pion couplings can now be determined to order $1/N_c$, by extrapolating the large N_c results to $N_c = 3$. Different extrapolations lead to different values for F/D , but the differences are of order $1/N_c^2$. The extraction of F/D for $N_c = 3$ from the large N_c results is tricky because the baryon $SU(3)$ representation is a function of N_c , a subtlety that was not present for $SU(2)$ representations. There are $1/N_c$ effects in the pion couplings because the baryon representations change with N_c , which leads to $1/N_c$ corrections in the Clebsch-Gordan coefficients of Tables 1 and 2. These “group-theoretic” $1/N_c$ representation corrections are completely calculable, and must be eliminated before one can extrapolate F/D to $N_c = 3$ with an accuracy of $1/N_c^2$. One extrapolation method is to equate the ratio

$$\frac{\Sigma^+ \rightarrow \Sigma^0 \pi^+}{\Sigma^+ \rightarrow \Lambda \pi^+} = 1 + \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (9.28)$$

computed in large N_c with its value in terms of D and F at $N_c = 3$,

$$\frac{\Sigma^+ \rightarrow \Sigma^0 \pi^+}{\Sigma^+ \rightarrow \Lambda \pi^+} = \frac{\sqrt{3}F}{D}, \quad (9.29)$$

which yields

$$\frac{F}{D} = \frac{1}{\sqrt{3}} + \mathcal{O}\left(\frac{1}{N_c^2}\right) = 0.58, \quad (9.30)$$

for $N_c = 3$. Another method is to recall that the non-relativistic quark model values for the ratios of pion couplings differ from large N_c QCD at order $1/N_c^2$. Thus the quark model prediction for F/D at $N_c = 3$ is accurate to $1/N_c^2$, which implies

$$\frac{F}{D} = \frac{2}{3} + \mathcal{O}\left(\frac{1}{N_c^2}\right) = 0.67. \quad (9.31)$$

A third method is to use the Skyrme model predictions [7] for the ratio of pion couplings at $N_c = 3$, which also differs from large N_c QCD at order $1/N_c^2$,

$$\frac{F}{D} = \frac{5}{9} + \mathcal{O}\left(\frac{1}{N_c^2}\right) = 0.56. \quad (9.32)$$

The difference of these three numbers is a consequence of $1/N_c^2$ effects; each determination yields an acceptable value for F/D at $N_c = 3$. The range of the three values implies an $\mathcal{O}(1/N_c^2)$ correction of about 0.1. In the following, we use the quark model value

$F/D = 2/3$. Any of the three values agrees well with the experimental value for the F/D ratio, 0.58 ± 0.04 [25]. The ratio of the decuplet-decuplet and decuplet-octet pion couplings to F and D are also determined to order $1/N_c$, and may be taken to be the non-relativistic quark model values. In the notation of ref. [26], the decuplet-octet pion coupling $\mathcal{C} = 2D$, and the decuplet-decuplet pion coupling $\mathcal{H} = 3D$. These results are known to agree with the experimental data [26][27].

One can also use ratios such as

$$\frac{p \rightarrow n\pi^+}{\Sigma^+ \rightarrow \Lambda\pi^+} \quad (9.33)$$

involving states in different K towers to determine the F/D ratio. In this case, it is important to remember that there are $1/N_c$ terms in the ratios of $g(K)$. These terms are purely group-theoretical in nature, and arise because the size of the weight diagram changes as a function of N_c . Since they are only group-theoretic, they are calculable (as was done in eq. (9.26)), and can be eliminated so that the extrapolation to $N_c = 3$ is valid to order $1/N_c$. This procedure yields a value for F/D at $N_c = 3$ which is consistent with eqs. (9.30)–(9.32) up to order $1/N_c^2$.

The ratio \mathcal{N}/\mathcal{M} is known to leading order in $1/N_c$, but it determines the ratios of the pion couplings up to order $1/N_c^2$, since the coefficients of \mathcal{N} in Table 1 for the pion couplings are suppressed by $1/N_c$ relative to those of \mathcal{M} . The leading order value for \mathcal{N}/\mathcal{M} also determines the leading order values for the kaon couplings. It is a straightforward exercise to verify that the kaon couplings are of order $1/\sqrt{N_c}$ relative to the pions, and are given by eq. (7.17). In the $SU(3)$ limit, one finds $c = -\sqrt{6} g(0)$ by comparing the ratio

$$\frac{p \rightarrow \Sigma^0 K^+}{p \rightarrow n\pi^+} = -\frac{1}{2\sqrt{N_c}} = \frac{c}{\sqrt{24N_c} g(0)}, \quad (9.34)$$

obtained using eq. (9.18) and the $SU(3)$ Clebsch-Gordan coefficients, and by using eqs. (6.6) and (7.17). The η -baryon couplings are of order $1/N_c$ relative to the pion-baryon couplings, and depend on the unknown $1/N_c$ term α in \mathcal{N}/\mathcal{M} in eq. (9.18). This lack of a prediction for the η couplings is related to the fact that the η couplings in eq. (8.6) depend on two operators at leading order in $1/N_c$. In the $SU(3)$ limit, one finds that

$$a = -\frac{4\sqrt{2}}{3}\alpha g(0), \quad b = -\frac{18\sqrt{2}}{3}g(0). \quad (9.35)$$

The above analysis considered baryons with finite strangeness in the large N_c limit, for which the matrix elements of pion, kaon and η axial currents are of order N_c , $\sqrt{N_c}$,

and 1, respectively. This analysis corresponds to working near the top of the $SU(3)$ weight diagram of fig. 7. For $N_c = 3$, however, the weight diagram contains only eight states, and the nucleon states are not far away from the other two corners of the weight diagram. One can therefore imagine other ways of extrapolating the large N_c results to $N_c = 3$. For example, one can instead consider states of definite U spin or definite V spin, and use these states to extrapolate from $N_c \rightarrow \infty$ to $N_c = 3$. For U spin, the K^0 , \bar{K}^0 and $-\pi^0/2 + \sqrt{3}\eta/2$ couplings are of order N_c , the K^+ , K^- , π^+ , π^- couplings are of order $\sqrt{N_c}$, and the $\sqrt{3}\pi^0/2 + \eta/2$ coupling is of order one. Nevertheless, the U spin or V spin extrapolations give the same F/D ratio as the above procedure, up to errors of order $1/N_c^2$.

9.1. Equal Spacing Rule for the Axial Couplings

The pion-baryon coupling $g(K)$ is linear in K at order $1/N_c$, as given in eq. (6.19). This result was derived with no assumption of $SU(3)$ symmetry, so it implies that $SU(3)$ breaking in the pion couplings is linear in K to order $1/N_c$. The $SU(3)$ breaking in the pion couplings can be extracted from the pionic decays of the decuplet baryons to octet baryons. The values of \mathcal{C} (defined in ref. [26]) for $\Delta \rightarrow N\pi$, $\Sigma^* \rightarrow \Lambda\pi$, $\Sigma^* \rightarrow \Sigma\pi$ and $\Xi^* \rightarrow \Xi\pi$ are 1.8, 1.5, 1.5 and 1.3, respectively [28], and should all be equal in the $SU(3)$ limit. The couplings clearly satisfy the linearity constraint on $SU(3)$ breaking. In particular, the $SU(3)$ breaking in the ratio $\Sigma^* \rightarrow \Lambda\pi/\Sigma^* \rightarrow \Sigma\pi$ is very small because both decays involve states in the $K = 1/2$ sector. Since pion couplings within a given K tower are determined to order $1/N_c$, the $SU(3)$ breaking in \mathcal{C} between $\Delta \rightarrow N\pi$ and $\Sigma^* \rightarrow \Lambda\pi$ must be the same as the $SU(3)$ breaking in the axial couplings for the beta decays $n \rightarrow pe^-\bar{\nu}$ and $\Sigma \rightarrow \Lambda e\nu$. This observation will allow a better determination of the F/D ratio from hyperon semileptonic decays, since $SU(3)$ breaking extracted from decuplet-octet baryon pion couplings can be subtracted from the beta decay couplings before performing the $SU(3)$ fit.

9.2. Magnetic Moments

The $SU(3)$ analysis of the baryon magnetic moments is similar to that of the axial currents. The stability of the baryon magnetic moments under renormalization, or equivalently, the large N_c behavior of pion-photoproduction, implies that at leading order, the baryon magnetic moments must be proportional to the axial currents [4]. This implies that the F/D ratio for the baryon magnetic moments is also $2/3$, in good agreement with the experimental value of 0.72. The difference between the experimental values for the F/D

ratios of the axial currents and magnetic moments is a $1/N_c^2$ correction. Thus the experimental data indicate that the $1/N_c^2$ correction is about 15–20%, which indicates that the $1/N_c$ expansion (at least for these quantities) is a reasonable expansion even for $N_c = 3$. Model independent relations for the baryon magnetic moments in the $1/N_c$ expansion are derived in ref. [29] in the context of the Skyrme model.

10. Baryon Masses

The $1/N_c$ expansion restricts the form of the baryon mass spectrum. In this section, we derive mass relations which are valid up to order $1/N_c$ without imposing $SU(3)$ symmetry. These mass relations constrain the form of $SU(3)$ breaking in the baryon mass spectrum.

The mass of a baryon in large N_c has an expansion of the form

$$M = N_c M_0 + M_1 + \frac{1}{N_c} M_2 + \dots, \quad (10.1)$$

where the leading contribution to the mass is order N_c , so that the baryon is infinitely heavy in the large N_c limit. Consistency conditions for the baryon masses can be derived from the scattering amplitudes considered in Sects. 6 and 7. Since baryon mass splittings are suppressed by powers of $1/N_c$ relative to the leading mass $N_c M_0$, it is possible to expand the baryon propagator about the static limit. The static baryon propagator is $i/k \cdot v$, where v is the baryon velocity and k is the residual momentum of the baryon. Baryon mass splittings change the baryon propagator to

$$\frac{i}{k \cdot v - \Delta M}, \quad (10.2)$$

where ΔM is the mass difference of the intermediate and initial baryons. In the rest frame of the baryon, the propagator reduces to $i/(\omega - \Delta M)$. Consistency conditions on the pion-baryon scattering amplitudes are valid for arbitrary pion energy, provided the energy is held fixed as $N_c \rightarrow \infty$. When the pion energy is greater than ΔM , it is possible to expand the propagator eq. (10.2) in a power series in $\Delta M/(k \cdot v)$. Since the terms in the expansion of the baryon propagator have different energy dependences, each term in the expansion of the propagator must separately satisfy the consistency conditions. The leading term in the expansion of the propagator is the lowest order propagator $i/k \cdot v$, which gives the consistency conditions for the pion couplings. The next term in the expansion of the propagator is $i\Delta M/(k \cdot v)^2$, which gives consistency conditions for baryon masses.

The pion-baryon scattering amplitude for $\pi + B \rightarrow \pi + B'$ is naively of order N_c , but the quark counting rules imply that it is at most of order one. This constraint was used to show that $[X, X]$ is at most of order $1/N_c$, using the lowest order propagator. The condition obtained by keeping the term proportional to ΔM in the expansion of the propagator is

$$[X^{jb}, [X^{ia}, M]] \lesssim \mathcal{O}\left(\frac{1}{N_c}\right), \quad (10.3)$$

which implies that

$$\begin{aligned} [X_0^{jb}, [X_0^{ia}, M_0]] &= 0, \\ [X_0^{jb}, [X_0^{ia}, M_1]] &= 0. \end{aligned} \quad (10.4)$$

Similarly the constraint that the amplitude $\pi + B \rightarrow \pi + \pi + B'$ be at most of order $1/\sqrt{N_c}$ gives the constraint

$$[X^{kc}, [X^{jb}, [X^{ia}, M]]] \lesssim \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (10.5)$$

which implies that M_2 satisfies

$$[X_0^{kc}, [X_0^{jb}, [X_0^{ia}, M_2]]] = 0. \quad (10.6)$$

A simpler (but equivalent) form of the above condition was derived in ref. [5] using chiral perturbation theory,

$$[X_0^{ia}, [X_0^{ia}, M_2]] = \text{constant}. \quad (10.7)$$

The solutions to eqs. (10.4) and (10.6) are that M_0 and M_1 are independent of I and J_{ud} , and M_2 is at most quadratic in I and J_{ud} . Arbitrary dependence on K and on J_s is allowed.

Further restrictions on the form of M are obtained by studying scattering amplitudes involving kaons, since these processes constrain the K and J_s dependence of M . Expanding out the intermediate propagator to first order in ΔM in $\pi + B \rightarrow K + B'$ and $\bar{K} + B \rightarrow K + B'$ scattering, and in $\bar{K} + B \rightarrow \pi + \pi + B'$, $\bar{K} + B \rightarrow K + \pi + B'$, and $\bar{K} + B \rightarrow K + K + B'$ scattering results in the constraints

$$[Y^{j\beta}, [X^{ia}, M]] \lesssim \mathcal{O}\left(\frac{1}{N_c}\right), \quad (10.8)$$

$$[Y^{j\beta}, [Y^{i\alpha}, M]] \lesssim \mathcal{O}\left(\frac{1}{N_c}\right), \quad (10.9)$$

and

$$[X^{kc}, [X^{jb}, [Y^{i\alpha}, M]]] \lesssim \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (10.10)$$

$$[X^{kc}, [Y^{j\beta}, [Y^{i\alpha}, M]]] \lesssim \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (10.11)$$

$$[Y^{k\gamma}, [Y^{j\beta}, [Y^{i\alpha}, M]]] \lesssim \mathcal{O}\left(\frac{1}{N_c^2}\right). \quad (10.12)$$

Eq. (10.9) restricts M_1 to be at most linear in K , while eq. (10.12) restricts M_2 to be at most quadratic in K . M_0 is independent of K .

The most general solution of eqs. (10.4)–(10.12) is that M has the form

$$M = N_c m_0 + m_1 K + \frac{1}{N_c} (m_2 I^2 + m_3 J^2 + m_4 K^2), \quad (10.13)$$

where m_i are constants independent of K which have an expansion in powers of $1/N_c$. (Note that the operator J_s^2 is equal to $K(K+1)$ and that $J \cdot J_s$ can be written as a linear combination of I^2 , J^2 and $K(K+1)$.) The baryon octet and decuplet each consist of four isospin multiplets. These eight baryon masses are parametrized by five mass parameters m_i in eq. (10.13), so there are three mass relations amongst the octet and decuplet masses which are valid up to corrections of order $1/N_c^2$. These relations are any three of the following four mass relations,

$$\frac{1}{3} (\Sigma + 2\Sigma^*) - \Lambda = \frac{2}{3} (\Delta - N), \quad (10.14)$$

$$\Sigma^* - \Sigma = \Xi^* - \Xi, \quad (10.15)$$

$$\frac{3}{4}\Lambda + \frac{1}{4}\Sigma - \frac{1}{2}(N + \Xi) = -\frac{1}{4}(\Omega - \Xi^* - \Sigma^* + \Delta), \quad (10.16)$$

$$\frac{1}{2}(\Sigma^* - \Delta) - (\Xi^* - \Sigma^*) + \frac{1}{2}(\Omega - \Xi^*) = 0, \quad (10.17)$$

each of which is valid including all terms of order $1/N_c$ in the baryon masses. *These relations are true irrespective of the mass of the strange quark, since they were derived without assuming $SU(3)$ symmetry.* We will refer to the linear combination of decuplet masses in eq. (10.16) as the equal spacing rule I, and the combination in eq. (10.17) as equal spacing rule II. Eq. (10.16) relates the violation of the Gell-Mann–Okubo formula to the violation of the equal spacing rule I for the decuplet. The Gell-Mann–Okubo formula and the equal spacing rule I are each violated at order $1/N_c$, but the difference of the two in eq. (10.16) is only violated at order $1/N_c^2$. The equal spacing rule II is only violated at order $1/N_c^2$.

One can characterize the deviations of the three $1/N_c$ mass relations by $1/N_c^2$ operators. Consistency conditions for M_3 are obtained from four meson-baryon scattering. There are three new operators which are first allowed at order $1/N_c^2$,

$$\frac{1}{N_c^2} M_3 = \frac{1}{N_c^2} (m_5 I^2 K + m_6 J^2 K + m_7 K^3) , \quad (10.18)$$

leaving no non-trivial relation amongst the eight octet and decuplet masses at order $1/N_c^2$. The eight masses can be used to determine the eight mass parameters m_0, \dots, m_7 . The corrections to eqs. (10.14)–(10.17) are $-(m_5 + m_6)/N_c^2$, $3m_6/2N_c^2$, $3(3m_5 - m_7)/16N_c^2$, and $-3(m_5 + m_7)/8N_c^2$, respectively.

One can also study the octet and decuplet relations at $N_c = 3$ starting from a $SU(3)$ symmetric Hamiltonian, and including a symmetry breaking term proportional to T^8 . In the symmetry limit, all the octet states are degenerate, and all the decuplet states are degenerate, so that eq. (10.14)–(10.17) are satisfied. The mass relations including terms of first order in symmetry breaking are the Gell-Mann–Okubo formula for the octet[†]

$$\frac{3}{4}\Lambda + \frac{1}{4}\Sigma - \frac{1}{2}(N + \Xi) = 0 , \quad (10.19)$$

and the equal spacing rule for the decuplet

$$\Omega - \Xi^* = \Xi^* - \Sigma^* = \Sigma^* - \Delta . \quad (10.20)$$

These relations are violated by non-analytic terms of the form $m_s^{3/2}$ and $m_s^2 \ln m_s$. The equal spacing rule II, however, has no $m_s^{3/2}$ and $m_s^2 \ln m_s$ corrections [30]. Deviations from the mass relations can be characterized by combining the expansion in the $SU(3)$ -breaking parameter m_s with the $1/N_c$ expansion,

$$\begin{aligned} \frac{1}{3}(\Sigma + 2\Sigma^*) - \Lambda &= \frac{2}{3}(\Delta - N) + \mathcal{O}\left(\frac{m_s}{N_c^2}\right) \quad (205 = 195) \\ \Sigma^* - \Sigma &= \Xi^* - \Xi + \mathcal{O}\left(\frac{m_s}{N_c^2}\right) \quad (191 = 215) \\ \frac{3}{4}\Lambda + \frac{1}{4}\Sigma - \frac{1}{2}(N + \Xi) &= -\frac{1}{4}(\Omega - \Xi^* - \Sigma^* + \Delta) + \mathcal{O}\left(\frac{m_s^{3/2}}{N_c^2}\right) \quad (6.5 = 3.4) \\ \frac{1}{2}(\Sigma^* - \Delta) + \frac{1}{2}(\Omega - \Xi^*) &= (\Xi^* - \Sigma^*) + \mathcal{O}\left(\frac{m_s^2}{N_c^2}\right) \quad (145.8 = 148.8) \\ \frac{3}{4}\Lambda + \frac{1}{4}\Sigma &= \frac{1}{2}(N + \Xi) + \mathcal{O}\left(\frac{m_s^{3/2}}{N_c}\right) \quad (1135 = 1128.5) \end{aligned} \quad (10.21)$$

[†] This formula is true for arbitrary N_c .

The experimental values for these relations (in MeV) are shown in parentheses.

The $1/N_c$ expansion helps explain the relative accuracies of the $SU(3)$ mass relations. Naively, equal spacing rule II should work to order m_s^2 . Since $SU(3)$ mass splittings in the baryons are of order 150 MeV, and each additional factor of m_s leads to a suppression by about 25%, one expects that the equal spacing rule II is satisfied to about 35 MeV. Instead, this relation works to about 3 MeV, since the violation of the equal spacing rule II is suppressed by an additional factor of $1/N_c^2$, which makes the relation work ten times better than naively expected. One can examine the other mass relations in a similar manner, and see that the $1/N_c$ expansion explains why some mass relations work much better than others. There is one relation that does not fit into this pattern, the Gell-Mann–Okubo formula, which works better than expected, and implies that $m_2 + m_4$ is small [24]. It works as well as equal spacing rule II, which is violated at $1/N_c^2$, whereas the Gell-Mann–Okubo formula is violated at order $1/N_c$. The equal spacing rule I also works much better than expected. However, this can be understood in the $1/N_c$ expansion, because eq. (10.16) relates the violation of the Gell-Mann–Okubo formula to the violation of the equal spacing rule I; if the Gell-Mann–Okubo formula works unexpectedly well, so must equal spacing rule I.

A similar analysis can be performed for the masses of baryons containing a single heavy quark. The spin-1/2 $SU(3)$ flavor $\bar{\mathbf{3}}$ and the spin-1/2 and spin-3/2 $\mathbf{6}$'s of heavy quark baryons contain eight isospin multiplets. At order $1/N_c$, these masses are parametrized by seven operators: 1 , K , K^2 , I^2 , J^2 , $J \cdot S_Q$ and $I \cdot S_Q$, where S_Q is the spin of the heavy quark Q . Thus, there is only a single mass relation at this order

$$\Sigma_Q^* - \Sigma_Q = \Xi_Q^* - \Xi_Q' + \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (10.22)$$

which is the analogue of eq. (10.15) for heavy quark baryons. In addition, there is a mass relation which relates the heavy quark baryon hyperfine splittings to the ordinary baryon hyperfine splittings [20][31],

$$\frac{1}{3} (\Sigma_Q + 2\Sigma_Q^*) - \Lambda_Q = \frac{2}{3} (\Delta - N) + \mathcal{O}\left(\frac{1}{N_c^2}\right). \quad (10.23)$$

11. $SU(3)$ Induced Representations

The previous sections have discussed the meson-baryon couplings using $SU(2)$ induced representations. This formulation is useful for understanding the implications of the $1/N_c$

expansion for $SU(3)$ breaking in the baryon sector. In the $SU(3)$ limit, one can combine the $SU(2)$ analysis with an $SU(3)$ tensor analysis, as was done in Sect. 10, to determine the meson-baryon couplings in the $SU(3)$ limit. In this section, we will determine the meson-baryon couplings in the $SU(3)$ limit by starting directly with an $SU(3)$ invariant formalism for the induced representations. Most of the analysis is identical to the $SU(2)$ analysis of Sect. 4, but there are several complications which occur in the $SU(3)$ analysis which are absent for $SU(2)$.

Let $|X_0^{iA}, \dots\rangle$ denote states which are the $SU(3)$ analogues of the states defined in eq. (3.1),[†]

$$(X_0^{iA})_{\text{op}} |X_0^{iA}, \dots\rangle = X_0^{iA} |X_0^{iA}, \dots\rangle. \quad (11.1)$$

The commutation relations

$$[J^i, X_0^{jB}] = i \epsilon_{ijk} X_0^{kB}, \quad [T^A, X_0^{jB}] = i f_{ABC} X_0^{jC} \quad (11.2)$$

imply that X_0 transforms as $(1, \mathbf{8})$ under $SU(2)_{\text{spin}} \otimes SU(3)_{\text{flavor}}$. For a finite spin \otimes flavor transformation (g, h) , X_0 transforms as

$$\begin{aligned} U_J(g)^\dagger X_0^{iA} U_J(g) &= D_{ij}^{(1)}(g) X_0^{jA}, \\ U_T(h)^\dagger X_0^{iA} U_T(h) &= D_{AB}^{(\mathbf{8})}(h) X_0^{jB}, \end{aligned} \quad (11.3)$$

where $D^{(1)}$ is a representation matrix in the adjoint representation of $SU(2)$, and $D^{(\mathbf{8})}$ is a representation matrix in the adjoint representation of $SU(3)$. The states $|X_0^{iA}, \dots\rangle$ are in one-to-one correspondence with the space of 3×8 matrices. The irreducible representations are orbits in the space of matrices under the transformations eq. (11.3). Large N_c baryons correspond to orbits which contain the element[†]

$$\overline{X}_0^{iA} = \begin{cases} 1, & i = A \text{ and } A \leq 3, \\ 0, & A > 3. \end{cases} \quad (11.4)$$

The little group of \overline{X}_0^{iA} is $SU(2) \times U(1) \times Z_2$, where $SU(2)$ is generated by $\mathbf{K} = \mathbf{I} + \mathbf{J}$, $U(1)$ is generated by T^8 , and Z_2 is generated by a 2π space rotation. The Z_2 factor splits the induced representations into fermionic and bosonic sectors (as for $SU(2)$), and will be

[†] We will use an uppercase flavor index $A = 1, \dots, 8$ for the $SU(3)$ X_0 's, and a lower case flavor index $a = 1, 2, 3$ for the $SU(2)$ X_0 's.

[†] This can be shown using the methods of Sect. 5.

omitted from now on. The ... in the state $|\overline{X}_0^{iA}, \dots\rangle$ is specified by the transformation properties of $|\overline{X}_0^{iA}, \dots\rangle$ under the little group $SU(2) \times U(1)$. The lowest lying baryons containing only u , d and s quarks are singlets under the $SU(2)$ group generated by \mathbf{K} , and have $U(1)$ charge $N_c/\sqrt{12}$. Let us define the states $|\overline{X}_0^{iA}, y\rangle$ by

$$\begin{aligned} \mathbf{K} |\overline{X}_0^{iA}, y\rangle &= 0, \\ T^8 |\overline{X}_0^{iA}, y\rangle &= N_c y |\overline{X}_0^{iA}, y\rangle, \end{aligned} \quad (11.5)$$

so that the physical baryons have $y = 1/\sqrt{12}$. Then one can define arbitrary states $|X_0^{iA}, y\rangle$ in the $SU(3)$ irreducible representation by applying spin and flavor transformations on $|\overline{X}_0^{iA}, y\rangle$. One can show that all matrices X_0^{iA} on the orbit of \overline{X}_0^{iA} in eq. (11.4) can be written as

$$X_0^{iA} = 2 \text{Tr } A T^i A^{-1} T^A, \quad (11.6)$$

where A is an $SU(3)$ matrix. Two A 's which differ by right multiplication by a hypercharge transformation, $A \rightarrow A e^{i\alpha T^8}$ give the same value for X_0^{iA} .[‡] One can define another operator on the $|X_0^{iA}, y\rangle$ basis states which commutes with X_0^{iA} ,

$$X_0^{8A} = 2 \text{Tr } A T^8 A^{-1} T^A, \quad (11.7)$$

which is well defined, and can be written directly in terms of X_0^{iA} as

$$X_0^{8A} = \frac{1}{\sqrt{3}} d_{ABC} X_0^{iB} X_0^{iC}. \quad (11.8)$$

The spin and flavor generators on the basis states are

$$\begin{aligned} J^i &= -i \epsilon_{ijk} X_0^{jA} \frac{\partial}{\partial X_0^{kA}}, \\ T^A &= -i f_{ABC} X_0^{iB} \frac{\partial}{\partial X_0^{iC}} + N_c y X_0^{8A}. \end{aligned} \quad (11.9)$$

The flavor generator T^A has two terms, a term which does not commute with X_0 and is of order one, and a term which commutes with X_0 and is of order N_c . This feature makes the N_c counting more complicated than for $SU(2)$, since T^A contains terms which grow with N_c . It is convenient to define

$$\hat{T}^A = -i f_{ABC} X_0^{iB} \frac{\partial}{\partial X_0^{iC}}, \quad (11.10)$$

[‡] This discussion closely parallels the quantization of the $SU(3)$ Skyrme model [32].

so that

$$T^A = \hat{T}^A + N_c y X_0^{8A}. \quad (11.11)$$

The states $|X_0^{iA}, y\rangle$ can be decomposed into states with definite spin and flavor quantum numbers. This analysis is identical to that in the $SU(3)$ Skyrme model, and will not be repeated here. The leading term X_0 for the meson-baryon couplings gives the axial current matrix elements (once an overall N_c is factored out) in the pion sector of order one, in the kaon sector of order $1/\sqrt{N_c}$, and in the η sector of order $1/N_c$. At order $1/N_c$, the consistency condition eq. (6.11) has a non-trivial solution for $SU(3)$, so that to order $1/N_c$,

$$X^{iA} = X_0^{iA} + \frac{\lambda'}{N_c} d_{ABC} X_0^{iB} T^C. \quad (11.12)$$

The d symbol vanishes for $SU(2)$, which is why this term was not found in Sect. 6. The correction in eq. (11.12), while formally of order $1/N_c$, actually makes a contribution of order one, since T^A has a piece that is of order N_c . Using the decomposition eq. (11.11) for T^A , one finds that

$$X^{iA} = X_0^{iA} + \frac{\lambda'}{N_c} d_{ABC} X_0^{iB} \left(\hat{T}^C + N_c y X_0^{8C} \right). \quad (11.13)$$

The identity

$$\sqrt{3} d_{ABC} X_0^{iB} X_0^{8C} = X_0^{iA} \quad (11.14)$$

shows that the order one piece is proportional to X_0 . Thus it is more convenient to write the $1/N_c$ correction to the axial coupling as

$$X^{iA} = X_0^{iA} + \frac{\lambda}{N_c} d_{ABC} X_0^{iB} \hat{T}^C. \quad (11.15)$$

This correction term yields an order $1/N_c$ correction to the \mathcal{N}/\mathcal{M} ratio in eq. (9.18). It produces a correction to the pion couplings at order $1/N_c^2$, to the kaon couplings at order $1/N_c$ and to the η couplings at order one (relative to the leading terms).

12. Chiral Loops

The $1/N_c$ expansion also allows one to compute corrections to the $SU(3)$ symmetry limit in a systematic way. The leading corrections are non-analytic corrections from chiral

perturbation theory.[†] Naively, these corrections grow with N_c because the pion-baryon couplings diverge like $\sqrt{N_c}$. We will see in this section, that the large N_c consistency conditions imply that the corrections to the axial currents decrease as $1/N_c$, instead of increasing as N_c . The $m_s^{3/2}$ corrections to the baryon masses are more interesting. The order $N_c m_s^{3/2}$ contribution to the baryon masses is $SU(3)$ singlet, the $m_s^{3/2}$ contribution is $SU(3)$ octet, and it is only the $m_s^{3/2}/N_c$ contribution that produces a correction to the $SU(3)$ mass-relations, such as the Gell-Mann–Okubo formula, which are derived under the assumption of octet symmetry breaking. This pattern confirms earlier suggestions [24][34][35][36] that the baryon masses might have a strong non-linear dependence on m_s , but that this non-linearity is such that it does not violate the Gell-Mann–Okubo formula. The chiral corrections to the baryon magnetic moments are more subtle, and will be discussed elsewhere.

12.1. Axial Currents

The leading non-analytic correction to the baryon axial currents is a $M^2 \ln M^2$ correction from the loop diagrams shown in fig. 8. The renormalization of X^{iA} is proportional to

$$N_c [X^{jC}, [X^{jB}, X^{iA}]] I^{BC}, \quad (12.1)$$

where the integral I^{BC} depends on the meson masses, and breaks $SU(3)$ symmetry. I^{BC} is equal to $M_\pi^2 \ln M_\pi^2/\mu^2$ for the pions, $M_K^2 \ln M_K^2/\mu^2$ for the kaons, and $M_\eta^2 \ln M_\eta^2/\mu^2$ for the η , and can be written as a linear combination of δ_{BC} , d_{BC8} and $d_{B88} d_{C88}$. Using the axial couplings to order $1/N_c$ given by eq. (11.15), one finds that

$$[X^{jC}, [X^{jB}, X^{iA}]] = \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (12.2)$$

so that the $M^2 \ln M^2$ correction is of order $1/N_c$. Thus meson-loop corrections in the baryon sector are suppressed by $1/N_c$, just as they are in the meson sector. It is important to keep in mind that the suppression in eq. (12.2) occurs only if one uses the large N_c definition of X , *i.e.* evaluating loop graphs including the complete large N_c tower of intermediate states, and using axial couplings with ratios determined consistently in large

[†] The η' mass is of order $1/\sqrt{N_c}$ [33], and η' loops should be included in the large N_c limit. The η' -nucleon coupling is of order $1/\sqrt{N_c}$, so η' loops contribute at order $1/N_c$, and are not important for the results discussed in this section.

N_c . This cancellation of the one-loop chiral logarithmic correction to the axial couplings is precisely what was found in earlier calculations for $N_c = 3$ [26].

For $SU(2)$ pion couplings, one has a stronger constraint on the chiral logarithmic correction than that of eq. (12.2). Because the order $1/N_c$ correction to X_0^{ia} must be proportional to X_0^{ia} , one has the constraint

$$[X^{jb}, [X^{jb}, X^{ia}]] \propto \frac{1}{N_c^2} X_0^{ia}, \quad (12.3)$$

which fixes the form of the $1/N_c^2$ term in the double commutator.

12.2. Baryon Masses

The leading non-analytic correction to the baryon masses is a M^3 correction from the graph in fig. 9. The loop graph is proportional to

$$N_c X^{iA} X^{iB} I^{AB} \quad (12.4)$$

where I^{AB} is equal to M_π^3 for the pions, M_K^3 for the kaons, and M_η^3 for the η , and can be written as a linear combination of δ_{AB} , d_{AB8} and $d_{A88} d_{B88}$. The form of the non-analytic correction can then be determined using eq. (11.15) for X^{iA} , and $SU(3)$ identities for the d -symbols.

A much simpler way of determining the form of the non-analytic corrections is to use the $SU(2)$ formalism of Sects. 6–9. The pion loop corrections are proportional to

$$N_c g(K)^2 X^{ia} X^{ia} M_\pi^3 = 3N_c g(K)^2 M_\pi^3 \quad (12.5)$$

using $X^{ia} X^{ia} = 3$ (note that X^{ia} is now an $SU(2)$ X). Using eq. (9.27) for $g(K)$, one finds that the order N_c term from the pion loops is a constant shift in the baryon mass, and the order one term (from the $1/N_c$ term in $g(K)$) is a correction proportional to the number of strange quarks, and so is an $SU(3)$ singlet plus octet. The kaon loop corrections are proportional to

$$c^2 (Y^{i\alpha} Y^{\dagger i\alpha} + Y^{\dagger i\alpha} Y^{i\alpha}) M_K^3. \quad (12.6)$$

The form of Y in eq. (7.14) shows that the order one kaon correction is of the form of a constant plus a term linear in K , and so is a $SU(3)$ singlet plus octet. The η loops are of order $1/N_c$. Thus the order N_c correction to the masses is $SU(3)$ singlet, the order one piece is singlet plus octet, and the first non-trivial correction first occurs at order $1/N_c$. A similar result can also be derived for the non-analytic chiral logarithmic correction to the baryon masses. The corrections had to have this form because the baryon mass formula eq. (10.13) was derived in Sect. 10 without assuming $SU(3)$ symmetry, and so it must be respected by the non-analytic corrections.

12.3. The Large N_c and Chiral Limits

Pion loop graphs such as the M_π^3 contribution to the nucleon mass shown in fig. 9 include the entire baryon tower as intermediate states. The baryon mass splittings ΔM are of order $1/N_c$. If one first takes the large N_c limit $N_c \rightarrow \infty$ and then the chiral limit $m_q \rightarrow 0$, the entire tower of baryons contributes to the non-analytic $m_q^{3/2}$ mass correction, whereas if one first takes the chiral limit $m_q \rightarrow 0$ and then the $N_c \rightarrow \infty$ limit, only the nucleon intermediate state contributes [37].

The non-commutativity of limits does not imply that there is a conflict between the large N_c and chiral expansions. Chiral perturbation theory is valid provided ΔM and M_π are small compared with $\Lambda_\chi \approx 1$ GeV, the scale of chiral symmetry breaking, irrespective of the value of $M_\pi/\Delta M$. The dependence of the nucleon mass on ΔM and M_π is calculable from the graph of fig. 9. The result is of the form

$$\frac{1}{16\pi f_\pi^2} M_\pi^3 F\left(\frac{M_\pi}{\Delta M}\right), \quad (12.7)$$

where $F(x)$ is known [28]. The function $F(x)$ has the correct limiting behavior as $x \rightarrow 0$ and $x \rightarrow \infty$ to correctly reproduce both the $(N_c \rightarrow \infty, m_q \rightarrow 0)$ and $(m_q \rightarrow 0, N_c \rightarrow \infty)$ limits. In the real world, $m_q \neq 0$ and $N_c \neq \infty$, and one should evaluate $F(x)$ at the physical value of $M_\pi/\Delta M$.

The $1/N_c$ expansion gives a systematic method of organizing the chiral corrections in the baryon sector, so that the non-analytic corrections are under control. Potentially large corrections either vanish, as for the axial currents, or can be reabsorbed into lower order terms in the Lagrangian, as for the masses. It will take a lot more work to see whether the $1/N_c$ expansion can be combined with baryon chiral perturbation theory to analyze the baryons properties in a systematic and controlled expansion.

13. Conclusions

The $1/N_c$ expansion provides a systematic expansion scheme for baryons. The contracted spin-flavor algebra for the baryon sector is sufficient to constrain the leading and subleading in $1/N_c$ contributions to various baryon static properties. The general expansion of operators such as X and the baryon masses is in powers of X , J/N_c , I/N_c (or \hat{T}/N_c for exact flavor $SU(3)$). The coefficients of the operators have an expansion in powers of

$1/N_c$ and K/N_c . In this work, we have shown that it is possible to consistently extend the $1/N_c$ expansion to the case of $N_f > 2$ light flavors even though the large N_c flavor representations are different from those for $N_c = 3$. The extension to more than two light flavors is more involved because of “representation effects”, which must be taken into account before determining the predictions for $N_c = 3$.

The form of the $1/N_c$ corrections shows that the $N_c \rightarrow \infty$ limit should be taken with I , J and K held fixed. There is no $1/N_c$ expansion for states with J of order N_c . Results for finite N_c are obtained by expanding about the $N_c \rightarrow \infty$ limit, and using the infinite tower of baryon states. All effects accounting for the fact that N_c is finite appear through $1/N_c$ suppressed operators. For instance, for finite N_c , the baryon tower terminates at $J = N_c/2$. The finite height of the tower away from $N_c \rightarrow \infty$ results in $1/N_c$ corrections to calculations performed with the infinite tower. These corrections are automatically included in the $1/N_c$ suppressed operators.

Many of the results obtained in the $1/N_c$ expansion for baryons are the same as those obtained in the Skyrme or non-relativistic quark models. The results obtained using the $1/N_c$ expansion are those model relations that work “well,” such as F/D ratios. The operator structure of the $1/N_c$ expansion is similar to that in the Skyrme and quark models. In large N_c QCD, the coefficients of the different operators are not determined by the consistency conditions. The models, on the other hand, make definite predictions for these coefficients. Model dependent relations which depend on the values of these coefficients, such as the absolute normalization of g_A , do not work well, and can not be derived from large N_c QCD. The non-relativistic quark model has a $SU(2N_f)$ spin-flavor symmetry for mesons as well as baryons. Only a contracted $SU(2N_f)$ spin-flavor symmetry for baryons exists in the $1/N_c$ expansion.

The $1/N_c$ expansion also provides a way of computing chiral loops in the baryon sector. Earlier computations of chiral loops were plagued by large non-analytic corrections [38][39]. The $1/N_c$ expansion provides an alternative calculational scheme in which the entire degenerate baryon tower is included as intermediate states in loop diagrams. This procedure leads to large cancellations, and makes the chiral expansion better behaved. Large corrections which do not cancel can be reabsorbed into lower order parameters.

Whether the $1/N_c$ expansion proves useful depends on the size of the $1/N_c$ corrections. The corrections appear to be under control for the baryon axial currents and masses. In the meson sector, there is one example where the $1/N_c$ corrections are large, the $\Delta I = 1/2$ rule [40]. In the large N_c limit, factorization is exact, and the $\Delta I = 1/2$ and $\Delta I = 3/2$

amplitudes are related by a Clebsch-Gordan coefficient, with no large enhancement of the $\Delta I = 1/2$ amplitude. However, other large N_c predictions such as Zweig's rule hold in the meson sector.

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Figure Captions

- Fig. 1. Graphs contributing to pion-baryon scattering at leading order in the $1/N_c$ expansion.
- Fig. 2. The leading diagram for the two point function of two axial currents in large N_c . Only the quark line is shown. An arbitrary number of internal gluons can be exchanged between the quark lines without changing the N_c -dependence of the diagram.
- Fig. 3. The dominant diagram for the n -point function in the large N_c limit. An arbitrary number of internal gluons can be exchanged.
- Fig. 4. A diagram contributing to the matrix element of the axial current between baryon states. The axial current can be inserted on any of the N_c quark lines.
- Fig. 5. The diagrams contributing to $\pi + B \rightarrow \pi + \pi + B'$ at leading order in $1/N_c$.
- Fig. 6. A diagram contributing to $\overline{K} + B \rightarrow \pi + B'$. The thick line is the s -quark.
- Fig. 7. The $SU(3)$ weight diagram for the spin-1/2 baryons for N_c colors. The long edge of the weight diagram has $(N_c - 1)$ states.
- Fig. 8. The diagrams for the one-loop correction to the baryon axial currents.
- Fig. 9. The diagram for the $m_s^{3/2}$ correction to the baryon masses.

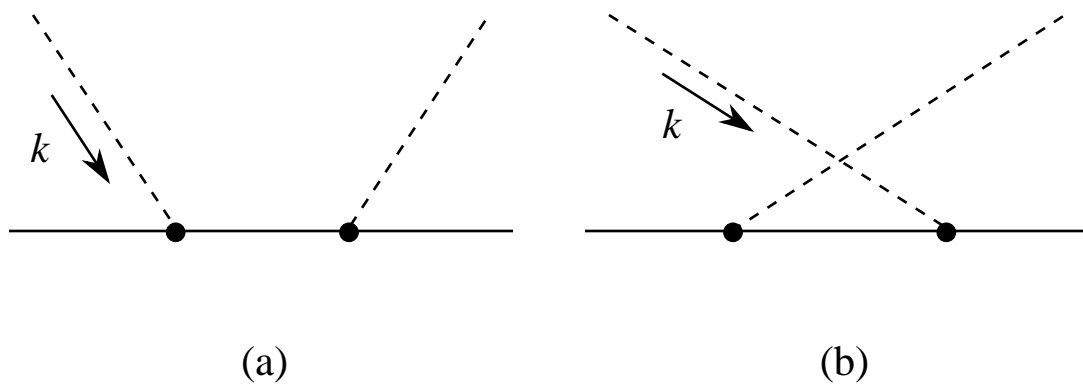


Figure 1

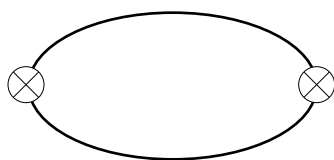


Figure 2

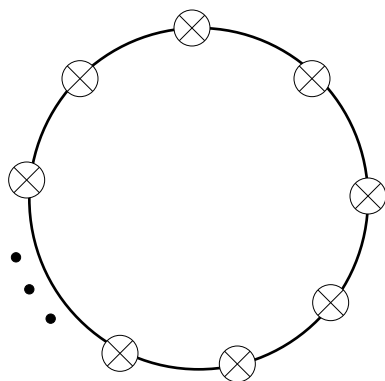


Figure 3

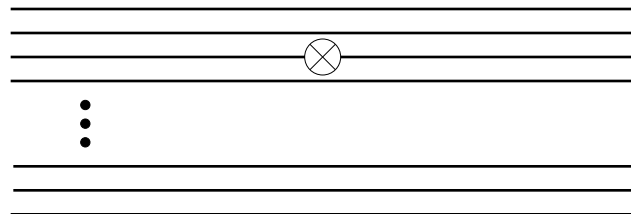


Figure 4

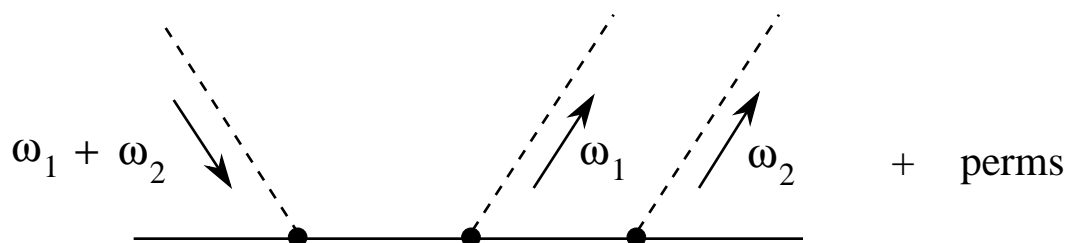


Figure 5

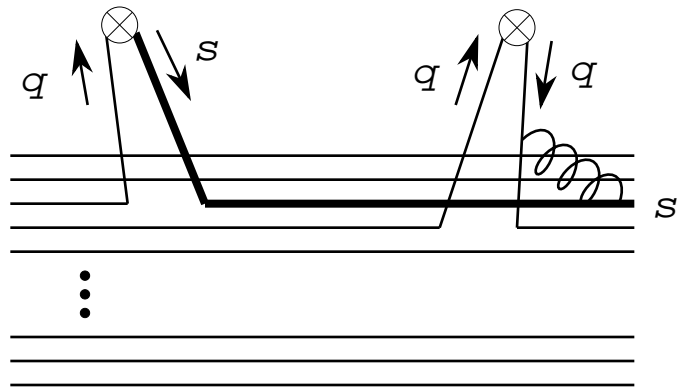


Figure 6


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      X   X
    X   XX  X
  X   XX  XX  X
••   XX  XX  XX  ••
X   XX  XX  XX  XX  X
X   XX  XX  XX  XX  XX  X
X   XX  XX  XX  XX  XX  XX  X
X   X   X   •••   X   X   X

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Figure 7

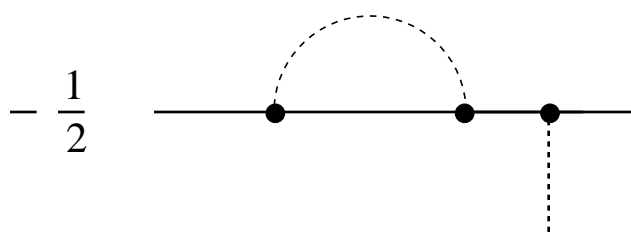
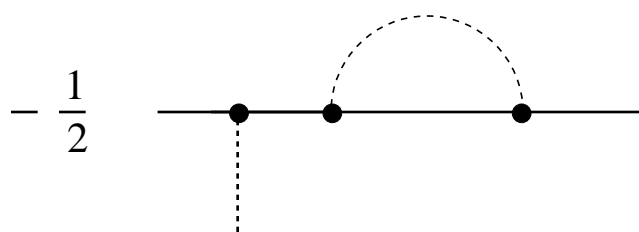
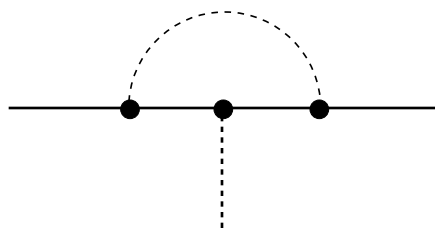


Figure 8

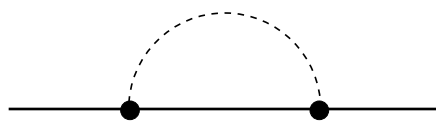


Figure 9